The moon subtends an angle of 1/110 radians in my field of view.

I wonder what Spot is thinking?

Not much, I'd guess. That's just what dogs do.
Who are those TOPS folks, anyway?

Ronald Jay Marson graduated from Seattle Pacific University with a B.S. in Chemistry, and from Harvard University with an MAT in Science Education. For 3 years he taught science and math, and supervised student teaching in West Africa as a Peace Corps volunteer. That’s where idealism first bumped into reality. With ingenuity and wit, Ron made do with limited resources, using local materials to teach his classes. Returning home, he refined his shoestring approach while teaching at a boarding school in Utah.

Ron’s rich and varied experience as an educator, his talent as a writer, respect for children, and love of teaching all come together in a science program he calls TOPS. As founder of TOPS Learning Systems, a nonprofit educational corporation, Ron is working hard to provide quality education based on resources available to everyone. His goal as an educator is to help new generations learn to love learning as they become their own best teachers. Ron stays refreshed running, biking and backpacking.

Peg Nazari Marson is a freelance artist and graphic designer with a colorful work history. In her “starving artist” days, she built considerable character while collecting a variety of skills working as a printer’s assistant, apple packer, bank teller, telephone operator, legal secretary, teacher’s aide, sign designer and woodworker, though not all at the same time. One of her favorite jobs was tutoring at-risk high school students in language arts, math and science. In 1979 she started her own commercial design business.

When she’s not busy drawing peoplets for TOPS, Peg works on her own art. She paints and draws in a variety of subjects, media and styles. She has had solo gallery shows, and consistently wins awards in juried exhibitions. Peg rounds out her life nurturing her organic garden and enjoying time with her grown daughter, Leah.

TOPS Learning Systems is a nonprofit educational corporation. Our purpose is to create the highest quality, friendliest, hands-on learning activities you’ll find anywhere, and to make them available everywhere, cheap! We offer over 40 books as wonderfully creative as this one, covering a wide range of topics in science, math and classroom management. Visit our online catalog at www.topscience.org, or write for our latest catalog.

TOPS Learning Systems 10970 S. Mulino Road Canby OR 97013
By my Kamal, all those objects subtend the same 1/25 rad angle!

That means this quarter is 25 diameters away...

...and so is this hula hoop!

...and this basketball is 25 diameters away...

SCIENCE WITH SIMPLE THINGS SERIES

Conceived and written by
Ron Marson

Illustrated by
Peg Marson

TOPS LEARNING SYSTEMS

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Canby OR 97013
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We at TOPS Learning Systems offer our warm thanks to these organizations for their faith in TOPS, and to the following individuals in the Department of Physics and Astronomy at SSU for their assistance and feedback during the development of this project:

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Why Teach TOPS?

Because your students have three brains – and multiple learning styles. They “do stuff” that’s engaging, confidence building, and, well, just plain fun. Creative use of everyday materials demystify science, make math processes concrete. Students learn to love learning as they become their own best teachers.

In a hands-on environment ...

Brain and muscle coordinate more efficiently as students manipulate simple materials to improvise, construct and create.

PSYCHO-MOTOR

Students develop their full range of intellectual capabilities, learning to observe, question, predict, test, analyze, synthesize, and communicate.

COGNITIVE

Students learn to learn ...

ACTIVITY

Activity-centered learning helps learners succeed at their own levels. Science becomes a creative, attractive and satisfying experience.

AFFECTIVE

students love to learn ...
Welcome, Dear Educator, …

... to PI in the SKY! This is our final book in a series of three which educate students about NASA’s 2007 GLAST space shot, while teaching standards-based math and science in engaging ways.

GLAST is an acronym for **Gamma-ray Large Area Space Telescope**. It will absorb, track and measure gamma ray photons across enormous energy ranges. Looking into the depths of solar flares, pulsars, quasars, active galactic nuclei, black holes, and gamma ray bursts, GLAST promises to help astronomers better understand interactions of matter at Nature’s most extreme and energetic levels.

Our first GLAST book, **FAR OUT MATH**, examines how logarithms make sense of huge ranges of astronomical data. Students fold paper into slide rules and use them for serious computing. Second-year algebra students grades 9-12, and the mathematically curious will gain a concrete understanding of logarithms and how they work — adding and subtracting exponent distances to multiply and divide corresponding numbers.

Our second GLAST book, **SCALE THE UNIVERSE**, explores distance and time over 40+ orders of magnitude, from protons to the limits of our universe, from gamma ray periods to the lifespan of red dwarfs. Science and math students grades 5-12 construct clever “Power of Ten” books, learning bunches about themselves and their place in the universe as they complete each page. Starting with what’s familiar (their bodies measured in meters, their rate of respiration measured in seconds), students reach both ways toward the infinitely large and the extremely small, measuring, ordering, plotting, and drawing.

Our third GLAST book, the one you’re reading now, teaches science and geometry students grades 5-12 about pi-radians and degrees, as a foundation for understanding subtended angles, apparent size, visual acuity and parallax. Student labs, ranging from simple to sophisticated, are organized into a flexible modular format to meet individual student needs and help teachers fit quality instruction into tight time schedules.

We want to know about your teaching experiences with these books. Please direct your comments and suggestions to TOPS at the address on our copyright page, and send a copy to Lynn Cominsky, our program director.

Happy sciencing,

Ron Marson

---

**How PI in the SKY ties in with National Science and Math Standards.**

**SCIENCE:**

Changes in systems can be quantified. Evidence for interactions and subsequent change and the formulation of scientific explanations are often clarified through quantitative distinctions – measurement. Mathematics is essential for accurately measuring change. (Constancy, Change, and Measurement Standard / p 118)

Students... should have the opportunity to use scientific inquiry and develop the ability to think and act in ways associated with inquiry, including asking questions, planning and conducting investigations, using appropriate tools and techniques to gather data, thinking critically and logically about relationships between evidence and explanations, constructing and analyzing alternative explanations, and communicating scientific arguments. (Science As Inquiry Standards / p 105)

It is important to maintain the spirit of inquiry by focusing teaching on questions that can be answered by using observational data, the knowledge base of science, and processes of reasoning.... (Content Standard D / p 189)

See also 5-8 Content Standards A, B, D, E & G; and 9-12 Content Standards A, B, E & G.

* (The National Science Education Standards)

---

**MATH:**

In grades 6-8 students will be able to: Use geometric models to represent and explain numerical and algebraic relationships; Use symbolic algebra to represent situations and solve problems, especially those that involve linear relationships; Develop and use formulas to determine the circumference of circles. Understand relationships among units and convert from one unit to another in the same system.

In grades 9-12 students will be able to: Analyze and connect characteristics of central angles, arc lengths and the lengths of radii; Use symbolic algebra to represent and explain mathematical relationships; Recognize the pattern in the relationship between angles measured in degrees and angles measured in radians; Use unit analysis to check measurement computations.

In grades 6-12 students will be able to formulate questions that can be addressed with data, and collect, organize, and display relevant data to answer them.

* (National Council of Teachers of Mathematics)
Getting Ready

- Decide which parts of this book you want to teach, then budget appropriate class time. See Overview on page 7 to understand various options for science and math classes. There are rich possibilities for independent study and extra credit for students who are motivated to do additional, or superlative, work.

- Photocopy relevant student materials as per guidelines on page 8. For your convenience, copying instructions are also summarized at the top of each page.

- Gather these simple materials. Here is a master list of everything you'll need to teach ALL the lessons. Consult the “materials box” at the bottom of each Student Lab for a list of items specific to each activity.

<table>
<thead>
<tr>
<th>Frequent Use</th>
<th>Occasional Or Single Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>✔ ruled notebook paper</td>
<td>✔ standard hole punch</td>
</tr>
<tr>
<td>✔ pencils with good erasers</td>
<td>✔ corrugated cardboard</td>
</tr>
<tr>
<td>✔ calculators (scientific calculators optional)</td>
<td>✔ current calendar with moon phases</td>
</tr>
<tr>
<td>✔ paper plates (generic, 9-inch diameter)</td>
<td>✔ small jars or cans</td>
</tr>
<tr>
<td>✔ scissors</td>
<td>✔ box or grocery bag (to carry stuff)</td>
</tr>
<tr>
<td>✔ string</td>
<td>✔ packaging tape</td>
</tr>
<tr>
<td>✔ masking tape</td>
<td>✔ broom</td>
</tr>
<tr>
<td>✔ clear tape</td>
<td>✔ clip board (for recording observations outside)</td>
</tr>
<tr>
<td>✔ meter sticks</td>
<td>✔ mirror (optional)</td>
</tr>
<tr>
<td>✔ index cards</td>
<td>✔ dark construction paper (black, dark blue)</td>
</tr>
</tbody>
</table>

- Organize a way to track assignments. It may be a good idea to keep student work on file in class. If you lack file space, substitute an empty copy paper box and brick. File folders and notebooks both make suitable assignment organizers. Students will feel a sense of accomplishment as their completed papers accumulate into an impressive portfolio. Since all assignments stay together, reference and review are easy. Ask students to tape a sheet of notebook paper inside the front covers of their folders or notebooks. Track individual progress by listing and initialing lesson numbers as they are completed.

- Communicate your grading expectations. We recommend that you grade on individual effort, attitude and overall achievement:
  - ✔ Effort: How many labs and how much written work has the student completed? Of what quality?
  - ✔ Attitude: Has the student worked to capacity, or wasted time? What evidence of personal initiative and responsibility?
  - ✔ Achievement: Assign tasks or ask questions that assess how well students have mastered key concepts.
Overview

Think of this book as a smorgasbord of science and math activities for grades 5-12. It contains six “buffet tables” (modules A-F), each arrayed with a variety of labs and learning tools. The beauty of a smorgasbord is that you don’t have to eat everything. You can sample this or that. Our goal is to provide teachers and students with flexibility and choice, across a wide grade-range of preference and taste.

Each “buffet table” is divided into student labs, student tools, and teaching notes. Page through this book to see how these selections repeat within each module. Scan the lab titles (these repeat in the teaching notes), to develop an overview of themes and possibilities.

You can select among student labs yourself, of course. Teachers traditionally assign tasks, and students dutifully comply (sometimes). But keep in mind that these TOPS labs have been lovingly crafted to speak to students directly, in language that they themselves can read and understand.

If you’ve ever wanted to abandon one-size-fits-all lesson plans and meet individual needs, here is an opportunity to experience student-directed teaching: Hand out the student labs, supply the simple materials, offer the demonstrators or introductions in the teaching notes when appropriate, and stand aside. Experience how it feels to respond to questions students ask you, rather than the other way around. Spend one class period, or up to a month of class periods (21 hours), with your students engaged in independent study.

Estimated times below are totals for all the labs in each module. Advanced students may skip many of these and proceed at a faster rate. Slower students may complete all labs within fewer modules. ALL students can potentially experience maximum learning at their individual ability levels in the time you have available. It doesn’t get more flexible than that!

MAP YOUR LABS: Modules A and B are conceptual prerequisites. Students who already understand pi might skip A and start with B. Advanced students who quickly grasp radians might complete just a couple of the labs before continuing to C, D or E. These three stand-alone modules may be attempted in any order or skipped altogether. Because a tool developed in E is also used in F, an arrow indicates this dependent relationship.

Module A: PI IS A CONSTANT (time: 2 hours)
The diameter of a circle fits into its circumference π times.

Module B: RADIANS AND DEGREES (time: 4 hours)
One radius of arc subtends a central angle of 1 radian (57.3°); π radians subtend π radians (180°); 2π radians subtend 2π radians full circle.

Module C: COUNTING DIAMETERS (time: 3½ hours)
Any object “n” diameters from your eye (n>5), subtends an angle of 1/n radians at your eye.

Module D: VISUAL ACUITY (time: 2½ hours)
A weather balloon is visible up to 7,000 balloon-diameters away. At this distance it subtends 1/7000 rad or about 30 arc seconds in your field of vision.

Module E: TIE YOUR KAMAL (time: 5½ hours)
Any object with an apparent size of 1/D radians, is D unit diameters from your eye.

Module F: LOOK FROM HERE AND THERE (time: 3½ hours)
Take Kamal readings. Use algebra to compute an object’s actual size; use parallax to compute actual distance. No need to go there!
**Photocopying**

Copy and staple reference sets of **Student Labs** for each module, A-F, that you plan to teach. For a class of 30 students, you might make 15 reference sets for students to share in pairs. Since students respond on their own paper, these may be reused in later class periods.

Each set of Labs begins with a page that orients students (and you) with a descriptive overview and a scope and sequence. Photocopy this “cover” page along with the labs to make student reference booklets. These booklets can have either single- or double-sided pages. Or, if you plan to use only a few Labs, you can simply reproduce those individually for student reference.

Remove pages from this book (see instructions below) for easier double-sided photocopying. Or keep this book intact by photocopying a set of single- (or double-) sided line masters.

Lab sets A-E have **Student Tools** printed on the reverse of the last page. These tools should be reproduced separately as single-sided copies. Students will cut or write on these to complete some labs. Notice that some Tools are designed to cut apart so two students can share one page.

**Teaching Notes** are labeled “Do not photocopy.” These pages are for your personal reference. If you plan to use them for a teacher workshop, you have our permission to photocopy **up to 3 pages**, as long as workshop participants promise not make copies of your copies. If you wish to copy more than 3 pages, please contact TOPS for permission.

<table>
<thead>
<tr>
<th>Photocopy Table</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Student Labs</strong></td>
</tr>
<tr>
<td>Make double-sided copies, one set for each pair of students in your largest class period. (Single-sided copies will also work.)</td>
</tr>
<tr>
<td><strong>A:</strong> 2 sheets; pgs 9/10 + 11/blank</td>
</tr>
<tr>
<td><strong>B:</strong> 3 sheets; pgs 19/20 + 21/22 + 23/blank</td>
</tr>
<tr>
<td><strong>C:</strong> 3 sheets; pgs 33/34 + 35/36 + 37/blank</td>
</tr>
<tr>
<td><strong>D:</strong> 2 sheets; pgs 49/50 + 51/blank</td>
</tr>
<tr>
<td><strong>E:</strong> 3 sheets; pgs 57/58 + 59/60 + 61/blank</td>
</tr>
<tr>
<td><strong>F:</strong> 2 sheets; pgs 71/72 + 73/74</td>
</tr>
<tr>
<td><strong>Student Tools</strong></td>
</tr>
<tr>
<td>Make single-sided copies, one for each student or pair of students of these pages:</td>
</tr>
<tr>
<td><strong>A:</strong> mm Beads/12, Pi Graph/13</td>
</tr>
<tr>
<td><strong>B:</strong> mm Beads/12 (reuse A’s), Snowman/24, Protractor/25</td>
</tr>
<tr>
<td><strong>C:</strong> Coin Ruler/38, Moon Ruler/39, Dime Ruler/40</td>
</tr>
<tr>
<td><strong>D:</strong> mm Dots/52</td>
</tr>
<tr>
<td><strong>E:</strong> Kamal Triangle/62, Planets To Scale/63</td>
</tr>
<tr>
<td><strong>F:</strong> Kamal Triangle/62 (use E’s construction)</td>
</tr>
<tr>
<td><strong>Teaching Notes</strong></td>
</tr>
<tr>
<td>No photocopies needed for modules A-F. These pages are for teacher reference only.</td>
</tr>
</tbody>
</table>

**Removing pages to photocopy:** Our pages are bound like a notepad. If the binding glue is light, they may pull off one at a time from the back. But if they threaten to tear, you’ll need to use a knife.

**One at a time:**

1. Bend the back cover all the way back and crease it (or tear it off).
2. Pull the exposed page free of the binding. Don’t pull so hard that it tears. The outer few pages are often glued more securely than the rest. If the glue is too strong, you’ll need to perform radical surgery (see next column).
3. Proceed to the next leaf until you remove all needed pages. (Don’t try to remove pages from the middle of the book first. This only works if the binding is ready to fall apart anyway.)

**Radical Surgery:**

1. Place a sharp knife between specific pages you wish to remove, with the cutting edge facing the binding, AND away from you. Close the book, and pull the knife through to cleanly divide the book into two parts.
2. Pull off pages one at a time, like sheets off a notepad.
3. Repeat step 1 as needed if pages threaten to tear. **Optionally,** use a sharp craft knife and a straight-edge to cut pages away from the binding.
Module A: PI IS A CONSTANT

What is this odd little number pi? It defines a fundamental property of all circles. Pi is the ratio C/D (or C/2R). And it is an irrational number (\(\pi = 3.14159\ldots\)).

It drove the Pythagoreans crazy that something as simple and basic as a circle could not be described using simple, whole numbers. But these labs won't drive you crazy. They're fun. Picture pi in your face! And they'll enhance your math rationality. By the time you finish with your pi, you'll know for sure that precisely \(\pi\) diameters (or \(2\pi\) radii) make a circle. Anything more or anything less is not perfectly round.

MAP YOUR LABS: A1 and A2 are "concept" labs that help you understand what pi means. Labs A3, A4, and A5, arranged farther down in the "map" below, deepen your understanding and apply concepts about pi. If these labs cover unfamiliar material, complete all five in order. As a review of what you've previously studied, A4 (and possibly A5) may be enough. Skip and choose to meet your needs.

A1: Concept
What's pi, anyway?
Calculate the ratio C/D for different sized circles. Lo and behold, you always get about 3.14!
(Time: 20 minutes)

A2: Concept
Chop circumferences into equal arc-lengths.
Every circumference divides into 3.14 diameters (\(\pi D\)) and 6.28 radii (\(2\pi R\)).
(Time: 15 minutes)

A3: Inquiry
Approximate pi using only a paper plate and string.
How many different ways can you show that \(\pi\) equals about 3.14?
(Time: 35 minutes)

A4: Extension
Make a pi graph.
Graph circumference as a function of diameter for circles large and small.
What's the slope?
(Time: 40 minutes)

A5: Application
Calculate Earth distances.
How much farther does it take GLAST to orbit the Earth than for you to walk around it?
(Time: 10 minutes)
Module A: PI IS A CONSTANT

A1: What's pi, anyway?

1. Get the Millimeter Beads supplement.
2. Count "beads" to find the circumference (C), diameter (D) and radius (R) for each big circle X, Y and Z. Record these measures (in mm) on your own notebook paper.

<table>
<thead>
<tr>
<th>circle X</th>
<th>circle Y</th>
<th>circle Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>C = 220 mm</td>
<td>D =</td>
<td>R =</td>
</tr>
</tbody>
</table>

3. Calculate the ratio C/D for each circle, rounded to two decimal places.
4. Pi is a constant. It is the same for all circles large and small.
   Describe pi (or π)...  
   a. in terms of C and D.  
   b. in terms of C and R.

concept: A1 A2 A3 A4 A5
materials: Millimeter Beads, calculator, your notebook paper (always).

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A2: Chop circumferences into equal arc-lengths.

1. On your Millimeter Beads page, divide the circumference of circle X into equal diameter lengths.  
   Label each curving diameter segment 1D, 2D, ....  
   a. How many times does its diameter fit into its circumference?  
      (Express your answer three ways: as a mixed number; as a decimal; in terms of pi.)
   b. Write an equation.  

2. Divide the circumference of circle Y into equal radius lengths.  
   Label each curving radius segment 1R, 2R, ....  
   a. How many times does its radius fit into its circumference?  
      (Express your answer three ways, as before.)
   b. Write an equation.

concept: A1 A2 A3 A4 A5
materials: Millimeter Beads, calculator, hand lens (optional)

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A3: Approximate pi using a paper plate and string.

1. Balance a paper plate level on the point of a pin.  
2. Poke the pin through at this balance point to mark the center.
3. How many different ways can you show that π ∼ 3 1/7?

inquiry: A1 A2 A3 A4 A5
materials: paper plate, pin, string, scissors, meter stick (or metric ruler), calculator

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**A4: Make a pi graph.**

1. Get the Pi Graph. Set up a graph at the left side with **Circumference** on the y-axis and **Diameter** on the x-axis.
2. Make a data table on the lower right side. Record the Circumference and Diameter of six items:
   - circle Z, circle Y, circle X,
   - a size-D battery,
   - a soda straw,
   - a dimensionless point.

Hints: Tie thread around the battery like a belt, then cut it open and measure it to find the circumference. Find diameter of a thin ring of the straw, then cut it open.

3. Plot and label each data point. Connect with a straight line.

**notes & vocabulary**

\[ \pi \text{ is the slope of this equation: } C = \pi D \]

4. Show that the slope of your graph line equals \( \pi \):

\[ \text{slope} = m = \frac{\text{rise}}{\text{run}}. \]

**extension:** A1 A2 A3 A4 A5

**materials:** Pi Graph page, Millimeter Bead page, thread, straightedge, size-D battery, straw, scissors

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**A5: Calculate Earth distances.**

Imagine walking around Earth’s equator. To make this journey, you’ve been granted the ability to walk on water and step through mountains.

1. You log 40,000 km from start to start. Use this information to plan your next mission impossible, a journey to the center of the Earth! How far is that?

2. NASA’s GLAST* mission orbits Earth at an altitude of 550 km and records powerful gamma rays. How much farther must GLAST travel than you did to complete one revolution? Include a diagram with your answer.

*Gamma-ray Large Area Space Telescope

**application:** A1 A2 A3 A4 A5

**materials:** calculator

© 2005 by TOPS Learning Systems. Contact: www.topscience.org
Millimeter beads

circle X
circle Z

circle Y

The diameter of each tiny bead equals one millimeter.
Module A: PI IS A CONSTANT

**A1: What's pi, anyway?**

1. Get the Millimeter Beads supplement.
2. Count "beads" to find the circumference (C), diameter (D) and radius (R) for each big circle X, Y and Z. Record these measures (in mm) on your own notebook paper.

![Image showing a character counting beads]

Each "bead" measures 1 mm across...

<table>
<thead>
<tr>
<th>circle X</th>
<th>circle Y</th>
<th>circle Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>C = 220 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D = 70 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R = 35 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C/D = 220 mm / 70 mm = 3.14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Calculate the ratio C/D for each circle, rounded to two decimal places.
4. Pi is a constant. It is the same for all circles large and small.
Describe pi (or π)...

- a. in terms of C and D.
- b. in terms of C and R.

**concept:** A1 A2 A3 A4 A5

**materials:** Millimeter Beads, calculator, your notebook paper (always).
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\[ \frac{C}{D} = ? \]

\[ \pi \]

**notes & vocabulary**

\[ \pi \text{ Pi is the number of times the diameter of a circle fits around the circumference.} \]

\[ \pi = 3.1416... \]

**INTRODUCTION**
Orient younger students to the Millimeter Beads page. If each "bead" is one mm across...

What is the radius of circle X? 35 mm
What is the arc length of quarter circle Z? 165 mm

Its diameter? twice that
Its circumference? four times that

**LESSON NOTES**

Bold numbers in these Lesson Notes and Model Answers always correspond to steps in the Student Labs above.

2. If you have provided non-consumable reference copies of these labs, remind students to keep them clean for other students' use. They should always answer on their own notebook paper, not on the photocopies.

3. Rounding to 2 decimal places means keeping 3 significant figures: 3.14.

Notice that this activity presents fractions in two forms: C/D and \( \frac{C}{D} \). On many pages throughout this book, the form we use is prompted by space limitations. Encourage students to write fractions as numerator over denominator when solving equations.

**MODEL ANSWERS**

2-3. Circle X

<table>
<thead>
<tr>
<th>C = 220 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>D = 70 mm</td>
</tr>
<tr>
<td>R = 35 mm</td>
</tr>
<tr>
<td>C/D = 220 mm / 70 mm = 3.14</td>
</tr>
</tbody>
</table>

Circle Y

<table>
<thead>
<tr>
<th>C = 440 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>D = 140 mm</td>
</tr>
<tr>
<td>R = 70 mm</td>
</tr>
<tr>
<td>C/D = 440 mm / 140 mm = 3.14</td>
</tr>
</tbody>
</table>

Circle Z

<table>
<thead>
<tr>
<th>C = 660 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>D = 210 mm</td>
</tr>
<tr>
<td>R = 105 mm</td>
</tr>
<tr>
<td>C/D = 660 mm / 210 mm = 3.14</td>
</tr>
</tbody>
</table>

4. Students might describe \( \pi \) in their own words, or in terms of a mathematical formula, or both:
4a. \( \pi \) is the ratio of a circle's circumference divided by its diameter. \( \pi = \frac{C}{D} \)
4b. \( \pi \) is the ratio of a circle's circumference divided by twice its radius. \( \pi = \frac{C}{2R} \).

Point out that all of these inputs (listed in order of increasing accuracy), approximate \( \pi: 3.14, 22/7, 3.1416, \pi \) key on a scientific calculator. Answers will differ slightly, of course, depending on which value for this non-repeating decimal you enter into your calculator.

**TEST FOR UNDERSTANDING**

A circle measures 7 mm at its widest and 22 mm around.

- a. What is its diameter? 7 mm
- b. What is its radius? 3.5 mm
- c. What is its circumference? 22 mm
- d. What's pi? \( \pi = \frac{C}{D} = 22/7 = 3.14 = \frac{C}{2R} \)

**ABOUT MATERIALS**

Notebook paper is always required, and assumed in all later activities. It goes without saying that a sharp pencil and good eraser are always essential tools.

Scientific calculators are wonderful, but basic four-operation calculators work just fine in all student labs. Input "3.14" or "3.1416" for calculators with no \( \pi \) key.
**A2: Chop circumferences into equal arc-lengths.**

1. On your Millimeter Beads page, divide the circumference of circle X into equal diameter lengths. Label each curving diameter segment 1D, 2D, ....
   
   a. How many times does its diameter fit into its circumference? (Express your answer three ways: as a mixed number; as a decimal; in terms of \( \pi \).)
   
   b. Write an equation. \( C = ? \)

2. Divide the circumference of circle Y into equal radius lengths. Label each curving radius segment 1R, 2R, ....
   
   a. How many times does its radius fit into its circumference? (Express your answer three ways, as before.)
   
   b. Write an equation.

---

**LESSON NOTES**

1. Students can use the same Millimeter Beads page in all lab activities in both modules A and B, adding information as they proceed. A new page may be started at any time for those who skip earlier activities.

**MODEL ANSWERS**

1a. The diameter fits \( 3\frac{1}{2} \) times (or) 3.14 times (or) \( \pi \) times around the circumference.

1b. \( C = \pi D \)

2a. The radius fits 6\( \frac{1}{2} \) times (or) 6.29 times (or) \( 2\pi \) times around the circumference.

2b. \( C = 2 \pi R \).

---

**TEST FOR UNDERSTANDING**

How many diameters make a circle?

- \( 3\frac{1}{2} \) diameters, 3.14 diameters, \( \pi \) diameters

How many radii make a circle?

- 6\( \frac{1}{2} \) radii, 6.29 radii, \( 2\pi \) radii

How many radii fit around a 180° half circle?

- 3\( \frac{1}{2} \) radii, 3.14 radii, \( \pi \) radii

---

**EXTENSION**

a. Use a compass to inscribe Circle X with a regular hexagon. Explain how you did this.

A regular inscribed hexagon divides the circumference of circle X into 6 equal parts:

\[
\frac{C}{6} = 220\; \text{mm}/6 = 36.6\; \text{mm} = 36.6\; \text{beads}
\]

Thus the hexagon meets the circumference at 36.6 mm intervals all around the circle.

b. Draw 1-radius arcs across Circle X from each vertex of the inscribed hexagon.

This results in the "flower mandala" at right, above.

C. Invent another mandala of your own design.

Here is one possibility. This one was inscribed with circles that have 2/3 the diameter of the "base" circle.
**A3: Approximate pi using a paper plate and string.**

1. Balance a paper plate level on the point of a pin.

2. Poke the pin through at this balance point to mark the center.

3. How many different ways can you show that \( \pi = 3.14 \)?

**Inquiry:** A1 A2 A3 A4 A5

**Materials:** paper plate, pin, string, scissors, meter stick (or metric ruler), calculator

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**Lesson Notes**
In the lower right corner of this lab (above materials), notice that this activity is defined as “inquiry.” Unlike A1 and A2 that address “concepts,” here students are invited to apply these concepts in an open-ended way. So leave them to their own devices. Don’t spoil it by telling them what to do.

**Model Answers**

3. We can think of five approaches. Encourage your students to come up with their own unique solutions.

- Cut a diameter-length of string and see how many times it fits around the circumference.
  (a little more than 3 times; 3 1/7 times; 3.14 times; \( \pi \) times)

- Cut a radius-length of string and see how many times it fits around the circumference.
  (somewhat more than 6 times, 6 2/7 times, 6.28 times, 2\( \pi \) times)
  (It may help to tape the diameter or radius string to the table in a curving arc, then count how many turns of the paper plate complete one revolution.)

- Cut a circumference-length of string and see how many diameters it makes.
  (a little more than 3 times, 3 1/7 times, 3.14 times, \( \pi \) times)

- Cut a circumference-length of string and see how many radii it makes.
  (somewhat more than 6 times, 6 2/7 times, 6.28 times, 2\( \pi \) times)

- Measure the circumference and diameter with a metric ruler and compute the ratios.
  \[ C = 72 \text{ cm} \]
  \[ D = 22.8 \text{ cm} \]
  \[ \frac{C}{D} = \frac{72 \text{ cm}}{22.8 \text{ cm}} = 3.16 \approx 3 \frac{1}{7} = \pi \]

**Test for Understanding**

Q: If it takes 22 minutes to walk around a crop circle, how long does it take to walk from the edge to its center?

A: It takes 22 minutes to walk 2\( \pi \) radii. To walk 1 radius to the center takes 22 min/2\( \pi \) = 3 1/2 min.
A4: Make a pi graph.

1. Get the Pi Graph. Set up a graph at the left side with Circumference on the y-axis and Diameter on the x-axis.

2. Make a data table on the lower right side. Record the Circumference and Diameter of six items:
   - circle Z, circle Y,
   - circle X,
   - a size-D battery,
   - a soda straw,
   - a dimensionless point.

Hints: Tie thread around the battery like a belt, then cut it open and measure it to find the circumference. Find diameter of a thin ring of the straw, then cut it open.

3. Plot and label each data point. Connect with a straight line.

4. Show that the slope of your graph line equals $\pi$:
   slope = m = rise/run.

**extension:** A1 A2 A3 A4 A5

**materials:** Pi Graph page, Millimeter Bead page, thread, straightedge, size-D battery, straw, scissors

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Module A: PI IS A CONSTANT

**A5: Calculate Earth distances.**

Imagine walking around Earth's equator. To make this journey, you've been granted the ability to walk on water and step through mountains.

1. You log 40,000 km from start to start. Use this information to plan your next mission impossible, a journey to the center of the Earth! How far is that?

2. NASA's GLAST* mission orbits Earth at an altitude of 550 km and records powerful gamma rays. How much farther must GLAST travel than you did to complete one revolution? Include a diagram with your answer.

*Gamma-ray Large Area Space Telescope

**application:** A1 A2 A3 A4 A5

**materials:** calculator

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---

**LEsson NOTES**

2. GLAST is an acronym that means Gamma-ray Large Area Space Telescope. Scheduled for launch into orbit around Earth in 2007, its mission is to "look" into space at the highest end of the electromagnetic spectrum to better understand high energy interactions in solar flares, pulsars, quasars, active galactic nuclei, black holes and gamma ray bursts. Explore http://glast.sonoma.edu for in-depth background.

---

**MODEL ANSWERS**

1. A pre-algebra solution:
   A journey straight to the center of the Earth is a trip of 1 radius. We know there are 2π radii in the 40,000 km circumference. So 40,000 km / 2π = 6,370 km.
   
   An algebraic solution:
   
   \[ C = 2\pi R \]
   \[ R = C / 2\pi = 6,370 \text{ km} \]

2. A pre-algebra solution:
   GLAST is 550 km farther from Earth's center than we are. Its distance is 550 km + 6,370 km = 6,920. We know that 2π radii fit the circumference of its orbit. So 6,920 km x 2π = 43,500 km. This figure is 3,500 km longer than the 40,000 km circuit at Earth's surface.

   An algebraic solution:

   \[ R_o = 550 + R_E \]
   \[ C_o = 2\pi R_o = 2\pi (550 + R_E) \]
   \[ C_o - C_E = 2\pi (550 + R_E) - C_E = 2\pi (550 + 6370) - 40,000 = 3,500 \text{ km} \]

However students express themselves, require a complete and logical presentation. The "right answer" without supporting equations and units is not good practice.

---

**TEST FOR UNDERSTANDING**

Q: Our moon has a circumference of 11,000 km. How far would you have to tunnel to reach its center?

A: Moon's radius = 11,000 km / 2π = 1,750 km

---

page 18
Module B: **RADIANS** and **DEGREES**

This arch looks a bit squashed, rather out-of-round.

I told Hector to use \( \pi \) stone slabs. Three aren't quite enough.

Why measure angles in radians when everyone uses degrees? Because radians are an incredibly useful tool for linking the **distance** of an object (from where you stand right now), to its **apparent size** in your field of vision.

More about this later. For now, do the math. Understand deeply, all the way to your bones, that \( \pi \) radii always subtend a half circle; that \( \pi \) radians equal 180°.

**MAP YOUR LABS:** If you understand \( \pi \) (module A), then you're ready for radians. Radians (or rads) measure angles, just as degrees do. If you've never heard of them, please start at the top with B1. Each new lab (extension, concept, ..., application, reinforcement), leads you deeper. But it is possible to skip and choose. If you already understand radians, challenge yourself with B6, then review B5.

**B1: Concept**
So what's a radian?
A paper plate, some simple measurements, a few snips... Aha! So that's a radian!
(Time: 45 minutes)

**B2: Extension**
Inscribe an equilateral triangle.
Lay a triangle with 60° corners on a circle. Does this match a radian angle?
(Time: 15 minutes)

**B3: Concept**
And what's a \( \pi \) radian?
With a flexible ruler, measure 1-rad radius arc lengths around a circle circumference. Soot a genuine \( \pi \) radian!
(Time: 30 minutes)

**B4: Concept**
Divide Circle Y into radian wedges.
Count millimeter beads to deduce that \( \pi \) radians of arc subtend a central angle of 180°. It all adds up.
(Time: 35 minutes)

**B5: Reinforcement**
Calibrate a protractor.
The relationship between degrees and radians is direct and obvious. Fractional \( \pi \) radians aren't quite so scary.
(Time: 55 minutes)

**B6: Application**
No protractors allowed!
These puzzles could be tough – unless you recall that 1 radius of arc subtends 1 radian;
that \( \pi \) rad subtend 180°.
(Time: 30 minutes)

**B7: Reinforcement**
Calibrate Circle Z in degrees and \( \pi \) radians.
This exercise is a good review of fractions. The angles are a bit weird, but everything adds up to \( \pi/2 \) rad or 90°.
(Time: 30 minutes)
Module B: RADIANS AND DEGREES

B1: So what's a radian?

1. Balance a paper plate upside down on a pin to find its center. Punch the pin through to mark this point.

2. Cut this plate into "radius wedges," so each slice has three equal sides (two straight plus one curved).

3. Use this protractor to estimate the size of each full wedge and the skinny remainder:
   a. Label the central angle of each wedge in degrees.
   b. Add up these angles. Do they total what you think they should? Explain.

c. How many radius wedges (how many radii), should fit around the circle? Why didn't you get that exact number?

concept: B1 B2 B3 B4 B5 B6 B7
materials: Paper plate, pin, scissors, index card, calculator.
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B2: Inscribe an equilateral triangle.

1. Get your Millimeter Beads page.
   a. Draw an equilateral (equal sided) triangle inside quarter circle Z, with its base sharing the "beaded" radius.

2. Subtend a 1-radian angle inside circle Z. Label it in both radians and degrees.
   b. Label its interior angles in degrees.

3. You've drawn two central angles in circle Z that are both subtended by one radius. So why aren't they of equal size?

notes & vocabulary
The angles in any triangle always add up to 180°.

extension: B1 B2 B3 B4 B5 B6 B7
materials: Millimeter Beads page, index card (or) metric ruler (or) compass tool, and a calculator
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Module B: RADIANS AND DEGREES

B3: And what's a pi radian?

1. Get a Snowman Circles page. Cut out the centimeter ruler.
   a. Roll a bit of masking tape sticky-side-out. Press this near the base of a battery.
   b. Prop the ruler upright by sticking the grey end onto the tape.

2. Measure one radius of arc around each Snowman Circle.
   a. Connect both endpoints to each circle’s center, to subtend central angles of 1 rad (one radian). Label each angle.
   b. Each radian wedge has a different area. So what’s equal about them?

3. Measure π radii of arc around each circle. (Begin where each radian slice ends.)
   a. Draw the central angles and label each π rad.
   b. What special angle do π radii always subtend? Write an equation.

4. Mark π/2 radii of arc on each circle. (Don’t overlap other arc lengths.)
   a. Label each central angle π/2 rad.
   b. What special angle is always subtended by π/2 radii? Write an equation.

notes & vocabulary
- Subtend: to be opposite; to extend from side to side.
- Arc: a curved section of a circle’s circumference.
- A π radius arc is 3.14 radii in length.

concept: B1 B2 B3 B4 B5 B6 B7 B8

materials: Snowman Circles, size-D battery, masking tape, scissors, index card or other straightedge, calculator

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B4: Divide Circle Y into radian wedges.

1. Get your Millimeter Beads page. Divide Circle Y into radius arc-lengths labeled 0R, 1R, 2R, ..., around the whole circle.

2. Connect each labeled “R point” to the circle’s center with a straight line. Including dashed lines, this circle should now be divided into 10 slices. (Count to be sure.)

3. Calculate and label the value of each central angle...
   a. To the nearest 1/7 rad.
      
      \[
      \text{Pi as a fraction: } \pi = \frac{22}{7}
      \]

   b. To the nearest tenth degree.
      
      \[
      \text{Unit conversion factor: } \pi \text{ rad} / 180^\circ = 1
      \]

4. Show that this circle’s quarters, halves, and wholes add up to the correct number of pi radians and degrees.

concept: B1 B2 B3 B4 B5 B6 B7

materials: Millimeter Beads page, straightedge, calculator.

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Module B: RADIANS AND DEGREES

B5: Calibrate a protractor.

1. Get the Protractor page.
   a. Poke a pinhole precisely through the bull’s-eye at the base of the leaning “flagpole.”
   b. Cut thread about 2 cm longer than this flagpole, and poke 1 cm of it into the pinhole. Tape it in place on the back.
   c. Position a tape “flag” on the thread to match the grey pattern.

   d. This prepares your protractor, featuring four semicircles: W, X, Y, and Z.

reinforcement: B1 B2 B3 B4 B5 B6 B7
materials: Protractor page, straight pin, thread, masking tape, scissors, calculator
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2. What are the radius and circumference, in centimeters, of semicircle W?
   a. How many radii are in this half circle?
   How many radians?
   b. What’s the difference between a radius and a radian?

3. What are the radius and circumference, in centimeters, of semicircle Z?
   a. Label Z in radians:
      0, 0.1, 0.2, 0.3, ..., 3.14.
   b. Mark these pi radians on Y:
      π, π/2, π/3, π/4.

4. Label semicircle X in degrees:
   (0°, 10°, 20°, 30°, ..., 180°).

5. Calibrate these π radians on semicircle Y:
   a: π/5, π/6, π/9, π/10, π/12, π/15, π/18, π/20, π/30, π/36, π/60, π/180.
   b: 5π/12, 9π/15, 2π/3, 3π/4, 5π/6, 8π/9, 19π/20.

B6: No protractors allowed!

Construct each angle in this lab following these guidelines:

1. Draw one radian with an index card and compass.
2. Draw 20 degrees with a flexible metric ruler and compass.
3. Draw one degree with only a metric ruler.
   (Hint: Start with a long straight radius of known length.)

application: B1 B2 B3 B4 B5 B6 B7
materials: index card, compass, a flexible Metric Ruler (or the paper ruler from activity B3 plus a long straightedge), calculator
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B7: Calibrate Quarter Circle Z in degrees and \( \pi \) radians.

1. Get the Millimeter Beads page.
   a. Divide the circumference of quarter circle Z into 11-millimeter intervals, from 0° to 90°.
   b. Label each interval in both degrees and \( \pi \) radians. Show your math.

2. Each one-millimeter bead subtends a tiny central angle. Express it six ways!
   a. Draw it.
   b. As a fractional radian with 1 in the numerator.
   c. As a fractional radian with \( \pi \) in the numerator.
   d. As a decimal radian.
   e. As a fraction of a degree.
   f. To the nearest arc minute.
   \( (1° = 60 \text{ arc minutes}) \)

reinforcement: B1 B2 B3 B4 B5 B6 B7
materials: Millimeter Beads page, calculator, hand lens (optional)

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Snowman Circles
Module B: RADIANS AND DEGREES

STUDENT TOOLS
(Photocopy class set, this side only.)

Protractor
Module B: RADIANS AND DEGREES

B1: So what’s a radian?

1. Balance a paper plate upside down on a pin to find its center. Punch the pin through to mark this point.

2. Cut this plate into "radius wedges," so each slice has three equal sides (two straight plus one curved).

   a. Label one of your wedges like this.

   b. How many radius wedges did you get? What fraction of a full wedge was left over?

   c. How many radius wedges (how many radii), should fit around the circle? Why didn’t you get that exact number?

3. Use this protractor to estimate the size of each full wedge and the skinny remainder:

   a. Label the central angle of each wedge in degrees.

   b. Add up these angles. Do they total what you think they should? Explain.

4. Each radius wedge defines a central angle of one radian.

   a. So what’s a radian? Write a definition in your own words.

   b. How many radians are in 360°? 180°?

   c. How many degrees are in 1 radian?

concept: B1 B2 B3 B4 B5 B6 B7
materials: Paper plate, pin, scissors, index card, calculator.

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LESSON NOTES

1. This center of mass is also the physical center of the plate only when its mass is evenly distributed. What happens if you tape a penny to the edge of the plate?

2. The plate remains most stable if students turn it upside down to draw radial divisions on its back surface. Lab partners work especially well here, since two hands are needed to bend both ends of the card to the curving circumference of the plate, while a partner marks off radial distances.

   Students who work alone can use bits of tape to lightly fix the plate to the table as they measure.

3. Students should avoid marking the protractor as a courtesy to others who will be using the same lab sheet later.

MODEL ANSWERS

2b. The paper plate divides into 6 radius wedges plus about \( \frac{2}{11} \) of a wedge left over. Expect fractional leftover wedges to vary widely due to errors in measurement.

2c. Since 2\( \pi \) radii fit into the circumference of a circle, it should theoretically divide into 6.28 wedges (slightly more than 6 \( \frac{1}{2} \) wedges). The difference is a result of measuring and cutting error, and because the plate has 3-dimensional contours.

3a. Check that students have labeled the central angle of all six radius wedges and the skinny remainder.

3b. \( 59° + 61° + 58° + 60° + 57° + 58° + 12° = 365° \). In this example the total is in excess of 360°, the actual number of degrees in a circle, due to measuring errors.

4a. A radian is the angle subtended by one radius of arc in any circle.

4b. 2\( \pi \) radians = 360°. So \( \pi \) radians = 180°.

4c. 1 radian \( \times 180°/\pi \) radians = 57.3°.

TEST FOR UNDERSTANDING

If you divided a pizza into radian slices, how many would you have? 2\( \pi \) slices = 6.28 slices
B2: Inscribe an equilateral triangle.

1. Get your Millimeter Beads page.
   a. Draw an equilateral (equal sided) triangle inside quarter circle Z, with its base sharing the "beaded" radius.
   b. Label its interior angles in degrees.

2. Subtend a 1-radian angle inside circle Z. Label it in both radians and degrees.
   Count the 1 mm beads to find one rad...

3. You've drawn two central angles in circle Z that are both subtended by one radius. So why aren't they of equal size?

---

LES ON NOTES

1a. Students who have worked with this Millimeter Beads page in Labs A, should continue using the same original page. Others who skipped these earlier labs can begin here with a fresh page.

There are many ways to inscribe the equilateral triangle. Allow students to ponder their own solutions.

_compass:_ Place its pivot at the point where the radius of mm beads meets the circumference. Span 1 radius, then swing it around to intersect the circumference. Draw 2 more radii from this intersection to each end of the original radius.

_index card:_ Mark a 1-radius length on the card's edge. Use this as a straightedge to draw 2 more lines of equal length, so they join on the circumference.

_metric ruler:_ Draw a 105 mm straight line from where the radius meets the circumference to where this length meets the circumference again. Connect this point to the center of the circle.

_math logic:_

Counting along the circumference:

- 165 mm beads = 90°
- 330 mm beads = 180°
- 110 mm beads = 60°

Count 110 mm beads along the circumference of the circle to determine a 60° central angle and draw it in. Join both radii to complete the equilateral triangle.

---

MODEL ANSWERS

1a and 2:

_1b._ Interior angles of an equilateral triangle add to 180°. So, 180°/3 equals 60° for each angle.

2. Count 105 beads (one radius), along the circumference to subtend a central angle of 1 radian. Convert radians to degrees by proportion, or by recalling that 180° = π radians:

\[
90°/165 mm = (x/105 mm)
\]

So, \( x = 57.3° \).

Or, \( (1 rad/1) \times (180°/π rad) = 57.3° \).

3. The straight radius subtends a larger central angle (60°) than the curving radius (57.3°).

TEST FOR UNDERSTANDING

Q: A regular polygon is inscribed in a circle. What can you say about the perimeters of these two figures?

A: The perimeter of a regular inscribed polygon is always less than the circumference of the circle that surrounds it. As its number of sides increases, its perimeter approaches the circle's circumference as a limit.
B3: And what's a pi radian?

1. Get a Snowman Circles page. Cut out the centimeter ruler.
   a. Roll a bit of masking tape sticky-side-out. Press this near the base of a battery.
   b. Prop the ruler upright by sticking the grey end onto the tape.

2. Measure one radius of arc around each Snowman Circle.
   a. Connect both endpoints to each circle's center, to subtend central angles of 1 rad (one radian). Label each angle.
   b. Each radian wedge has a different area. So what's equal about them?

3. Measure \( \pi \) radii of arc around each circle. (Begin where each radian slice ends.)
   a. Draw the central angles and label each \( \pi \) rad.
   b. What special angle do \( \pi \) radii always subtend? Write an equation.

4. Mark \( \pi/2 \) radii of arc on each circle. (Don't overlap other arc lengths.)
   a. Label each central angle \( \pi/2 \) rad.
   b. What special angle is always subtended by \( \pi/2 \) radii? Write an equation.

\[
\frac{\pi}{2} = \text{equation}
\]

concept: B1 B2 B3 B4 B5 B6 B7 B8

materials: Snowman Circles, size-D battery, masking tape, scissors, index card or other straigntedge, calculator

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INTRODUCTION

Draw a semicircle on your board or overhead.

\[\pi \text{ rad} = 180^\circ\]

Describe it in terms of radii (a distance):
3.14 radii subtend a straight angle.
\( \pi \) radii subtend a straight angle.
\( \pi \) radii subtend 180°.

Describe it in terms of radians or rad (an angle):
\( \pi \) radii equal a straight line.
\( \pi \) rad = 180°.

LESSON NOTES

1. For precise alignment, hold the ruler perpendicular to and flush with the table as you push the end of it onto the taped battery.
2. The battery anchors the “zero” end of the ruler. This leaves one hand free to curve the ruler around the circle. And the other hand is free to mark the endpoints with a pencil.
3. Students may draw a blank when they read “\( \pi \) radii.” Well, \( \pi \) is a constant, a number that equals 3.1416..., or 3.14 rounded off. And radii is plural for radius. So \( \pi \) radii requires this arc length: 1 radius + 1 radius + 1 radius + 0.14 radius.

MODEL ANSWERS

1-3. Angles may assume different orientations. This "snowman" is reduced to fit the page:

2b. The wedge in each circle has a different area, but all have a central angle equal to 1 rad.
3b. In any circle, a \( \pi \) rad arc length (3.14 R) always subtends a straight line (\( \pi \) rad = 180°).
4b. In any circle, a \( \pi/2 \) rad arc length (1.57 R) always subtends a right angle (\( \pi/2 \) rad = 90°).

TEST FOR UNDERSTANDING

How much is a \( \pi \) radian slice of pizza? Half a pizza.
B4: Divide Circle Y into radian wedges.

1. Get your Millimeter Beads page. Divide Circle Y into radius arc-lengths labeled 0R, 1R, 2R, ..., around the whole circle.

2. Connect each labeled "R point" to the circle's center with a straight line. Including dashed lines, this circle should now be divided into 10 slices. (Count to be sure.)

3. Calculate and label the value of each central angle...
   a. To the nearest 1/7 rad.
   b. To the nearest tenth degree.

4. Show that this circle’s quarters, halves, and wholes add up to the correct number of pi radians and degrees.

MODEL ANSWERS

1-3.

4. 1st quadrant:
   \[
   \frac{7}{7} \text{ rad} + \frac{1}{7} \text{ rad} = \frac{11}{7} \text{ rad} = \frac{\pi}{2} \text{ rad}
   \]
   \[57.3^\circ + 32.7^\circ = 90.0^\circ\]

2nd quadrant:
   \[
   \frac{3}{7} \text{ rad} + \frac{7}{7} \text{ rad} + \frac{1}{7} \text{ rad} = \frac{11}{7} \text{ rad} = \frac{\pi}{2} \text{ rad}
   \]
   \[24.6^\circ + 57.3^\circ + 8.2^\circ = 90.1^\circ = 90^\circ\]

3rd quadrant:
   \[
   \frac{6}{7} \text{ rad} + \frac{5}{7} \text{ rad} = \frac{11}{7} \text{ rad} = \frac{\pi}{2} \text{ rad}
   \]
   \[49.1^\circ + 40.9^\circ = 90.0^\circ\]

4th quadrant:
   \[
   \frac{2}{7} \text{ rad} + \frac{7}{7} \text{ rad} + \frac{2}{7} \text{ rad} = \frac{11}{7} \text{ rad} = \frac{\pi}{2} \text{ rad}
   \]
   \[16.4^\circ + 57.3^\circ + 16.4^\circ = 90.1^\circ = 90^\circ\]

   top half:
   \[
   \frac{\pi}{2} \text{ rad} + \frac{\pi}{2} \text{ rad} = \pi \text{ rad}
   \]
   \[90^\circ + 90^\circ = 180^\circ\]

   bottom half:
   \[
   \frac{\pi}{2} \text{ rad} + \frac{\pi}{2} \text{ rad} = \pi \text{ rad}
   \]
   \[90^\circ + 90^\circ = 180^\circ\]

   whole:
   \[
   \pi \text{ rad} + \pi \text{ rad} = 2\pi \text{ rad}
   \]
   \[180^\circ + 180^\circ = 360^\circ\]

TEST FOR UNDERSTANDING

Q: If you cut a 1 radian wedge from a slice of pizza, how much is left? Answer in both radians and degrees.

A: whole pizza: \(2\pi \text{ rad} = 360^\circ\)

   1 rad wedge: \(1 \text{ rad } \times 180^\circ/\pi \text{ rad} = 57.3^\circ\)

   remains: \((2\pi - 1) \text{ rad } \times 180^\circ/\pi \text{ rad} = 57.3^\circ\)

   check: \(5.28 \text{ rad } \times 180^\circ/\pi \text{ rad} = 302.6^\circ\)
**B5: Calibrate a protractor.**

1. Get the Protractor page.
   a. Poke a pinhole precisely through the bull’s-eye at the base of the leaning “flagpole.”
   b. Cut thread about 2 cm longer than this flagpole, and poke 1 cm of it into the pinhole. Tape it in place on the back.
   c. Position a tape “flag” on the thread to match the grey pattern.
   d. This prepares your protractor, featuring four semicircles: W, X, Y, and Z.

**reinforcement:** B1 B2 B3 B4 B5 B6 B7  
**materials:** Protractor page, straight pin, thread, masking tape, scissors, calculator  
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**INTRODUCTION**

How many radians is \( \pi \) radians? 3.14 radians; the number of rads in a semicircle.

**MODEL ANSWERS**

2. \( W_{\text{RADIUS}} = 7 \text{ cm}; \ W_{\text{CIRCUMFERENCE}} = 22 \text{ cm}. \) (2a) \( 3\frac{1}{2} \) radii; \( 3\frac{1}{4} \) radians.
(2b) A radius is a distance while a radian is the subtended angle.
3. \( Z_{\text{RADIUS}} = 10 \text{ cm}; \ Z_{\text{CIRCUMFERENCE}} = 31.41 \text{ cm}. \) (3a, b) Having just calibrated semicircle Z from 0 to 3.14 radians, students will understand that: \( \pi \) rad = 3.14 rad, or all of Z; \( \frac{\pi}{4} \) rad = 1.57 rad; \( \frac{\pi}{2} \) rad = 1.05 rad; \( \frac{\pi}{4} \) rad = 0.79 rad.
4-5. Having just calibrated semicircle X from 0 to 180°, students see that \( \pi \) rad = 180°, or all of Z. So they can calculate fractional parts of Z by dividing into a much friendlier form of \( \pi \) (180° instead of 3.14 rad).
   5a. \( 180\frac{2}{6} = 36°; \ 180\frac{4}{6} = 30°; \) and so on.
   5b. \( (5)(180)/12 = 75°; \)
   (9)(180)/15 = 108°;
   and so on.

**TEST FOR UNDERSTANDING**

**Q:** Express the central angle in a semicircle in two measures.
**A:** \( \pi \) radians and 180°

**TEACHING NOTES**

1a. Threading a pinhole may be difficult for some. Call on students with good eyesight and steady hands to assist others.
1d. Semicircle Z has a radius of 10 cm. So each 1 cm of circumference equals \( \frac{1}{10} \) of its radius, and subtends \( \frac{1}{10} \) radian.
Module B: RADIANS AND DEGREES

B6: No protractors allowed!

Construct each angle in this lab following these guidelines:

1. Draw one radian with an index card and compass.
2. Draw 20 degrees with a flexible metric ruler and compass.
3. Draw one degree with only a metric ruler.
   (Hint: Start with a long straight radius of known length.)

application: B1 B2 B3 B4 B5 B6 B7
materials: index card, compass, a flexible Metric Ruler (or the paper ruler from activity B3 plus a long straightedge), calculator

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MODEL ANSWERS

1. Draw a circle with the compass and mark the precise length of its radius against the edge of an index card. Bend this radius length around the circle to define an arc length of 1 radian. Connect each end to subtend the central angle.

\[
1 \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = 57.3^\circ
\]

2. Convert 20° to decimal radians:

\[
20^\circ \times \pi \text{ rad} / 180^\circ = \frac{\pi}{9} \text{ rad} = 0.349 \text{ rad}.
\]

Use a compass to draw a segment of a circle with a radius precisely 10 cm long. Measure precisely 3.49 cm of arc length (0.349 rad) around its circumference with the flexible ruler. This arc length subtends a central angle of 20°.

3. Convert 1° to decimal radians:

\[
1^\circ \times \pi \text{ rad} / 180^\circ = \frac{\pi}{180^\circ} = 0.0175 \text{ rad}.
\]

Draw a long line of known length, say 20 cm, across the full sheet of paper. Let this be the radius of a circle. Calculate the fraction of this radius that subtends 1°.

\[
0.0175 \times 20 \text{ cm} = 0.35 \text{ cm} = 3.5 \text{ mm}.
\]

This very short arc on such a large circle can be treated as a short perpendicular line segment. So the 1° angle is well represented by a long, skinny right triangle, with the long base equal to 20 cm and a short rise of 3.5 mm.

TEST FOR UNDERSTANDING

Q: Knowing there are 60 arc seconds in 1°, how would you draw a 1 arc second angle similar to what you have just drawn?

A: Draw a skinny triangle 1/60 as tall:

3.5 mm / 60 = 0.06 mm. This height is 60 micrometers, about as thick as a human hair!
Module B: RADIANS AND DEGREES

B7: Calibrate Quarter Circle Z in degrees and \( \pi \) radians.

1. Get the Millimeter Beads page.
   a. Divide the circumference of quarter circle Z into 11-millimeter intervals, from 0\(^\circ\) to 90\(^\circ\).
   b. Label each interval in both degrees and \( \pi \) radians. Show your math.

   reinforcement: B1 B2 B3 B4 B5 B6 B7
   materials: Millimeter Beads page, calculator, hand lens (optional)

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LESSON NOTES
1. The outside perimeter of quarter circle Z must be free of stray calculations or other pencil marks from earlier use. Erase as necessary to provide a clean margin space.

MODEL ANSWERS
1. Dividing the circumference into 11 mm intervals divides the 90\(^\circ\) span into 15 intervals.
   Thus, 90\(^\circ\)/15 = 6\(^\circ\) per interval.
   And, 6\(^\circ\) x \( \pi \) rad/180\(^\circ\) = \( \pi \)/30 rad per interval.
2a. See “skinny” angle, lower left.
2b. \( R = 105 \) mm, so 1 mm subtends 1/105 rad.
2c. \( \pi /2 \) rad = 165 mm, so 1 mm = \( \pi /330 \) rad
2d. 1/105 rad = \( \pi /330 \) rad = .0095 rad
2e. 11 mm = 6\(^\circ\), so 1 mm = (6/11)\(^\circ\)
2f. (6/11)\(^\circ\) x 60 arc minutes /1\(^\circ\) = 33 arc minutes

Note: 33 arc minutes closely approximates the apparent size of both our Sun and moon as seen from Earth. See lab C7, page 47.

TEST FOR UNDERSTANDING
Q: Express 45\(^\circ\) on circle Z in terms of mm beads and pi radians.
A: 90\(^\circ\) = 165 beads = \( \pi /2 \) rad
Dividing by 2,
45\(^\circ\) = 82.5 beads = \( \pi /4 \) rad
Imagine looking out from the center of a circle to a clock face on the circle’s circumference.

If the clock is “n” diameters from your eye (n > 5), then it subtends 1/n radians in your field of view. Counting intervening clock-diameters (or lengths, widths, heights for other objects), becomes a simple, accurate way to estimate apparent angular size.

MAP YOUR LABS:

Obey the prerequisite arrows! You must complete toolmaking lab C1 before continuing with labs C2-C5.

Notice that labs C6-C8 form a stand-alone cluster with access through “gate” C6.

C1: Toolmaking
Make a coin-diameter ruler.
Notice how it measures distance in diameters and angles in radians for pennies, nickels and quarters.
(Time: 20 minutes)

C2: Concept
Observe from the plane of your table top.
See how a penny looks half as tall when placed twice the number of penny diameters from your eye.
(Time: 15 minutes)

C3: Concept
Arrange coins to have the same apparent size.
Observe how a penny, nickel and quarter appear to be the same size when placed equal coin diameters from your eye.
(Time: 15 minutes)

C4: Inquiry
Explore angular size relationships among coins.
Understand that any object “n” diameters from your eye subtends 1/n radians in your field of view (n > 5).
(Time: 40 minutes)

C5: Inquiry
Make a ruler from paper plates.
Correlate nickel diameters on your Coin Ruler with plate diameters on your Plate Ruler.
(Time: 40 minutes)

C6: Modeling
Does a Moon Ruler correctly scale the real thing?
Make a Moon Ruler. Correlate its size with our real Earth and Moon and the distance between.
(Time: 20 minutes)

C7: Application
Gaze at a paper plate moon.
Use your knowledge of radians and subtended angles to make it equal in angular size to your paper-punch moon.
(Time: 30 minutes)
C1: Make a coin-diameter ruler.

1. Get the Coin Rulers page.
   a. Carefully trim off the outside margins. Cut along the middle dashed line to divide the page into equal halves of 4 strips each.
   b. Tape these halves end to end, so all lines match and you spell the word “pennies.”

2. Gently fold between the 4 strips (precisely on the solid black lines) to make a long “tent.”
   a. Sharply crease the 2 folds between the white strips. (Don’t crease the third fold.)
   b. Tuck the grey strip inside the tent. The folds should keep it in shape without tape.

3. Examine your Coin Ruler.
   a. In what units does it measure distance?
   b. In what two units does it measure angles?
   c. Curving circumferences are matched to flat coins. Does this work better up close or far away? Explain.

C2: Observe from the plane of your table top.

1. Stand 2 pennies on your table in tiny balls of clay. Push them down to touch the tabletop.

2. Extend the Coin Ruler about 1 finger-width forward from the table edge. Stand the pennies 10 and 20 diameters along its length. (Be sure to measure along the PENNIES side.)

3. Kneel at the table to position the pupil of one eye at “zero point,” where radian angles converge.
   a. Compare penny heights (as seen through one eye from zero point).
   b. What angle does each penny subtend at the pupil of your eye?

4. Experiment with other distances. If one penny appears twice as tall as another...
   a. What’s true about their relative distances to your eye?
   b. What’s true about the angles they subtend in your visual field?

concept: C1 C2 C3 C4 C5 C6 C7 C8
materials: 2 pennies, a bit of clay, a Coin Ruler

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C3: Arrange coins to have the same apparent size.

1. Using your Coin Ruler, decide where to line up a penny, nickel and quarter so they all appear to have the same diameter when viewed from “zero point.” Make a prediction.
   a. Check your prediction.
   b. Explain your results in terms of subtended angles.

2. Repeat the experiment, this time alternating between looking at the coins through one eye and both eyes.
   a. How does a two-eyed view look different than a one-eyed view?
   b. Subtended angles remain the same whether looking at the coins through one eye or two. So what else influences apparent size?

concept: C1 C2 C3 C4 C5 C6 C7 C8
materials: 3 pennies, 1 nickel, 1 quarter, a bit of clay, a Coin Ruler

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C4: Explore angular size relationships among coins.

1. Experiment with 3 pennies and your Coin Ruler. Make them appear taller or shorter than each other (2x, 3x, 4x, 5x...).

   a. Predict how the coins will look before you view them.
   b. Describe your results.

2. Repeat your experiment using a penny, a nickel, and a quarter (in any order) instead of 3 pennies. Did you get the same result?

3. As long as you measure distance in “diameters,” does the actual size of an object matter in determining angular size? Explain.

   attention!
   Don’t measure pennies in quarter diameters! Be sure to use the correct side of the ruler.

4. Arrange coins in other interesting perspectives. Invent challenges for your classmates to solve.

inquiry: C1 C2 C3 C4 C5 C6 C7 C8
materials: 3 pennies, 1 nickel, 1 quarter, a bit of clay, a Coin Ruler

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**Module C: COUNTING DIAMETERS**

**C5: Make a ruler from paper plates.**

1. Tape a paper plate over dark background paper on your wall. Extend a Paper-Plate Ruler as follows:
   - **a.** Line up 25 paper plates on the floor. Alternate face-up and face-down.
   - **b.** Join them (up-rim touching down-rim) with masking tape.
   - **c.** Number the plates boldly from 1 to 25 from the wall.

2. Get your Coin Ruler.
   - **a.** Hold the zero end of the **nickel** scale facing up, directly under one eye.
     (Pinch the scale from underneath to hold it in one hand.)
   - **b.** Hold a nickel in your other hand.

3. Experiment by making the nickel “eclipse” the paper plate.
   - **a.** Compare nickel diameters on the Coin Ruler with plate diameters on the floor.
   - **b.** Report your findings in terms of subtended angles.

**Inquiry:**
- **C1** C2 C3 C4 C5 C6 C7 C8
- **materials:** black construction paper (optional), paper plates, masking tape, Coin Ruler, a nickel

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**C6: Does a Moon Ruler correctly scale the real thing?**

1. Get half of a Moon Ruler page. Make a ruler that counts paper-punch diameters as follows:
   - **a.** Cut out the three numbered strips, and tape them in order. Use the edge of your desk as a guide.
   - **b.** Fold the ends as marked. Punch out the “moon” hole with a hole punch.

**modeling:** C1 C2 C3 C4 C5 C6 C7 C8

**materials:** half of the Moon Ruler page, scissors, clear tape, hole punch

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2. Our moon (diameter = 3,480 km) orbits Earth (diameter = 12,740 km) at an average distance of 384,000 km. Show that your Moon Ruler accurately scales these distances:
   - **a.** Do the correct number of moons fit across Earth? Show your math.
   - **b.** Do the correct number of moons fit in between? Show your math.

3. What angular size is the moon as seen from Earth? How many degrees is this?
C7: Gaze at a paper plate moon.

1. Examine your Moon Ruler. Calculate how far from a paper plate you must stand so it looks as big as the real moon. Answer in both paper-plate diameters and in meters.

2. Cut a piece of string 10.0 meters long. Wrap it around a corrugated cardboard “spool.”

3. Tape a plate to a tree or wall outside. Measuring with string and a meter stick, find your calculated distance.

4. View the plate through your Moon Ruler from that distance. What do you observe?
   (If your arms are short, crease the “moon” end of the ruler lengthwise to keep it straight.)

**application:** C1 C2 C3 C4 C5 C6 C7 C8

**materials:** Moon Ruler, scissors, a paper plate, a meter stick, string, masking tape, a piece of corrugated cardboard

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C8: Test your Moon Ruler on family and friends.

1. Tape a shiny dime to a wall or window to represent the moon. Ask your family and friends to stand far enough away so this dime looks as big as the real moon.

2. Get the Dime Ruler page and assemble the ruler. Use it to measure each guess in radians from eye to dime.

3. First chance you get, show them the real moon outside, looking through the Moon Ruler.
   a. Write about your interactions.
   b. By what factors did they overestimate its size?

4. Are these results typical? If time allows, extend your research. Would you get the same results with a compact disk (CD) moon disk?

**notes & vocabulary**

Consult a calendar to find the current phase of the moon. You might spot it in the daytime sky if you know where to look:

- **A first quarter moon** follows the sun across the sky: about 90° behind.

- **A last quarter moon** leads the sun across the sky: about 90° ahead.

**research:** C1 C2 C3 C4 C5 C6 C7 C8

**materials:** dime, Dime Ruler page, Moon Ruler, clear tape, current calendar with moon phases

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page 37
Moon Ruler

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Module C: COUNTING DIAMETERS
STUDENT TOOLS

(Photocopy one page per two students, this side only.)
Module C: COUNTING DIAMETERS

C1: Make a coin-diameter ruler.

1. Get the Coin Rulers page.
   a. Carefully trim off the outside margins. Cut along the middle dashed line to divide the page into equal halves of 4 strips each.
   b. Tape these halves end to end, so all lines match and you spell the word “pennies.”

2. Gently fold between the 4 strips (precisely on the solid black lines) to make a long “tent.”
   a. Sharply crease the 2 folds between the white strips. (Don’t crease the third fold.)
   b. Tuck the grey strip inside the tent. The folds should keep it in shape without tape.

3. Examine your Coin Ruler.
   a. In what units does it measure distance?
   b. In what two units does it measure angles?
   c. Curving circumferences are matched to flat coins. Does this work better up close or far away? Explain.

toolmaking: C1 C2 C3 C4 C5 C6 C7 C8
materials: Coin Rulers page, scissors, clear tape

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INTRODUCTION
When students have constructed their Coin Rulers, give them this primer to its “hows and whys.”

There are three main features to notice: the calibrations along the ruler’s length; the protractor at the leading end that measures subtended angles; and the viewpoint (lower left) from which these angles can be observed.

The vertical lines divide the ruler into “coin diameters”: penny diameters on the penny scale, nickel and quarter diameters on the other faces. There are two lines at each calibration, one straight, one curved. The bolder curved lines are segments of concentric circles that subtend radian angles at their common central viewpoint. The viewing distance (the number of coin diameters) determines the radius of each circle.

a. Place a Coin Ruler on the table, “quarters” facing your students, and stand a quarter in a bit of clay perpendicular to the first calibrated line. What angle does this coin subtend from the viewpoint? The flat coin subtends 45°. But if it were curved to match the darker line, it would subtend 1 radian, or 57.3°.

b. How far from the viewpoint must a quarter stand for the “straight” and “curved” lines to subtend nearly equal angles? (Beyond about 5 coin diameters, the curves are moderate enough to subtend nearly the same angles as the straight lines.)

c. What angle does a quarter subtend when viewed at these diameter distances:
   7 diameters? $\frac{1}{2}$ rad
   20 diameters? $\frac{1}{20}$ rad
   1 diameter? Less than 1 rad, only 45°

LESSON NOTES
2. This 3-D “tent” flattens into a 4-layer strip that folds in half for compact storage.

MODEL ANSWERS
3a. The Coin Ruler measures distance in coin diameters. Distance is calibrated in pennies, nickels or quarters along its three sides.
3b. The ruler measures angles in radians and degrees.
3c. Up close, the ruler doesn’t measure subtended angles of flat coins accurately. At greater distances than about 5 coins ($\frac{1}{5}$ rad), however, a flat coin and a shallow curving arc of the same length subtend nearly equal angles.

TEST FOR UNDERSTANDING
Which is longer, a straight row of 14 quarters, 15 nickels, or 16 pennies? 14 quarters.

What angle does a penny that’s 10 diameters from your eye subtend in your visual field? $\frac{1}{10}$ rad = 6°

What angle does a nickel that’s 10 diameters from your eye subtend in your visual field? $\frac{1}{10}$ rad = 6°

What angle does a quarter that’s 10 diameters from your eye subtend in your visual field? $\frac{1}{10}$ rad = 6°
Module C: COUNTING DIAMETERS

C2: Observe from the plane of your table top.

1. Stand 2 pennies on your table in tiny balls of clay. Push them down to touch the tabletop.

2. Extend the Coin Ruler about 1 finger-width forward from the table edge. Stand the pennies 10 and 20 diameters along its length. (Be sure to measure along the PENNIES side.)

3. Kneel at the table to position the pupil of one eye at “zero point,” where radian angles converge.
   a. Compare penny heights (as seen through one eye from zero point).
   b. What angle does each penny subtend at the pupil of your eye?

4. Experiment with other distances. If one penny appears twice as tall as another... a. What’s true about their relative distances to your eye? b. What’s true about the angles they subtend in your visual field?

concept: C1 C2 C3 C4 C5 C6 C7 C8
materials: 2 pennies, a bit of clay, a Coin Ruler

INTRODUCTION

Look across the surface of a book with one eye, holding the pupil of your eye in the plane of the cover. What do you see? A straight horizontal line.

LESSON NOTES

1-3. Students should use the penny scale to measure penny diameters (not the nickel or quarter scales). If they forget here, there is no penalty, because any scale enables students to simply double distances, which halves apparent angular size. Not so in the next activity, where mixing scales will confuse results.

Students tend to use too much clay. Even a BB-sized pinch will do. Make sure they sink coins firmly into the clay so rims touch the table top. These clay “feet” should support the coins, but not add to their heights.

From zero point, the table surface “disappears” and the pennies appear to stand on one horizontal line, as in the introduction. This zero point starts at the observer’s eye, about 1 penny diameter (or finger width) beyond the table edge. To make this allowance, hang the ruler over the table edge by this difference.

Notice that C1 is underlined in the lower right corner of the lab above). This signals, here and elsewhere, that the underlined lab is a prerequisite to the current lab, indicated in bold.

MODEL ANSWERS

3a. The near penny appears twice as tall as the far penny.
3b. The near penny subtends 1/10 rad at the pupil of the eye. The far penny subtends 1/20 rad.
4a. If one penny appears 2 times taller than another, it is half the distance to your eye. This is true for pennies at 8 and 16 diameters, 10 and 20 diameters, 12 and 24 diameters, and so on. This pattern breaks down over short distances. See Introduction to Lab C1.
4b. If one penny appears 2 times taller than another, it subtends twice the angle in your visual field. This is true for pennies at 1/6 and 1/16 rad, 1/10 and 1/20 rad, 1/12 and 1/24 rad, and so on. This pattern breaks down for angles greater than about 1/5 radian. A penny just 1 diameter from the eye, for example, doesn’t subtend the expected angle of 1/1 rad or 57.3°, rather only 45°.

TEST FOR UNDERSTANDING

A penny stands 16 penny-diameters from your eye. Where should you place a second penny so it appears...
   a. Two times taller? 8 diameters away.
   b. Half as tall? 32 diameters away.
C3: Arrange coins to have the same apparent size.

1. Using your Coin Ruler, decide where to line up a penny, nickel and quarter so they all appear to have the same diameter when viewed from “zero point.” Make a prediction.
   a. Check your prediction.
   b. Explain your results in terms of subtended angles.

2. Repeat the experiment, this time alternating between looking at the coins through one eye and both eyes.
   a. How does a two-eyed view look different than a one-eyed view?
   b. Subtended angles remain the same whether looking at the coins through one eye or two. So what else influences apparent size?

concept: C1 C2 C3 C4 C5 C6 C7 C8
materials: 3 pennies, 1 nickel, 1 quarter, a bit of clay, a Coin Ruler

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INTRODUCTION

Try this simple experiment with your students to demonstrate “psychological” size compensation:

a. Ask students to hold a penny in one hand at reading distance, and a quarter in the other hand at least twice that distance from their faces. Viewed through one eye, which coin looks larger? When these coins are viewed together through one eye, the quarter still appears larger.

b. Maintaining these relative distances in each hand, eclipse the quarter by passing the penny in front of it. Viewed through one eye, which coin has the greater angular size? The penny has greater angular size, because it fully eclipses the quarter with room to spare.

c. Why did the quarter look bigger even though it had a smaller angular size? Our previous experience with pennies and quarters tells us the quarter is larger, even though it actually subtended a smaller angle in our visual field.

LESSON NOTES

Students with confused results have likely confused scales. Penny diameters, for example, are only valid for placing pennies, not nickels or quarters.

MODEL ANSWERS

1a. Project the zero end of the ruler about 1 finger-width over the table edge to approximate the eye’s true position behind the table. Then stand the penny 20 penny-diameters away, the nickel 20 nickel-diameters away, and the quarter 20-quarter diameters away as marked on the Coin Ruler. In these positions, the 3 coins appear to have equal size. The number “20” of course, is arbitrary. Any number greater than 5 will do. The penny just eclipses the nickel, which just eclipses the quarter. Moving the eye a little right or left so the nickel and quarter come into view, all coins have the same apparent height.

1b. Each coin is precisely 20 of its own diameters from the eye. All 3 coins subtend equal angles (1/20 rad) in my field of view.

2a. With both eyes open, the coins suddenly take on their “normal” relative sizes, with the penny looking smallest and the quarter looking largest. Close one eye, and the coins seem to “snap” back to equal size, where the front coin just covers the coin behind.

2b. Opening both eyes allows us to see all 3 coins in a more dimensional perspective. Since our brain knows that quarters are largest and pennies smallest, it compensates by “seeing” what it thinks it should. So the 3 coins look relatively larger and smaller, even though they equally subtend 1/20 rad in my field of vision.

TEST FOR UNDERSTANDING

When the Moon eclipses the Sun, both bodies appear to have equal size. What can you conclude about their relative distances from Earth? They are an equal number of diameters away. If the Moon is "n" Moon diameters from Earth, then the Sun is "n" Sun diameters away.

Do these bodies have equal sizes psychologically? Opinions may be divided on this. The Sun dominates the daytime sky with its extraordinary brightness. But the Moon rules the night, and looms large in our psyches.
C4: Explore angular size relationships among coins.

1. Experiment with 3 pennies and your Coin Ruler. Make them appear taller or shorter than each other (2x, 3x, 4x, 5x...).
   
   a. Predict how the coins will look before you view them.
   b. Describe your results.

2. Repeat your experiment using a penny, a nickel, and a quarter (in any order) instead of 3 pennies. Did you get the same result?

3. As long as you measure distance in "diameters," does the actual size of an object matter in determining angular size? Explain.

4. Arrange coins in other interesting perspectives. Invent challenges for your classmates to solve.

**Inquiry:** C1 C2 C3 C4 C5 C6 C7 C8

**Materials:** 3 pennies, 1 nickel, 1 quarter, a bit of clay, a Coin Ruler

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**Lesson Notes**

1. Allow students to invent their own experiments in true inquiry mode. Here are some ideas for kids who get stuck:
   - Make a penny 3 times taller than a second and 9 times taller than a third.
   - Place coins at 5, 15, and 45 penny diameters. (To measure 45 penny diameters, use the coin ruler twice.)
   - Make 1 penny appear as wide as 2 touching side by side.
   - Two pennies at 20 diameters have the same width as 1 penny at 10 diameters.

2-3. Repeating the experiment in step 1 with different coins drives home the conclusion in step 3.

4. Possible challenges:
   - Make a quarter appear 3 times shorter than a penny, then 3 times taller.
   - Place the penny at 6 diameters and the quarter at 18 diameters. Then switch coins.
   - Make 3 pennies touching side by side appear as wide as 1 nickel.
   - Place a nickel at 6 diameters and three pennies at 18. Works for 3 nickels and 1 penny, too.
   - Make a quarter appear the same size as a paper plate.
   - Tape the paper plate to a wall, 6 plate diameters beyond a quarter placed at 6 coin diameters. This experiment anticipates C5.

**Model Answers**

1a. One prediction among many possibilities: A penny positioned 5 diameters from the eye should appear twice as tall as a penny at 10 diameters, and four times as tall as a penny at 20 diameters.

1b. Looking through one eye, in line with the table's surface, the pennies appear as predicted. From nearest to farthest, they subtend angles of 1/6 rad, 1/10 rad, and 1/20 rad respectively. These angles are in the same 1:2:4 ratio as the apparent size of each coin.

2. This 1:2:4 ratio of coin diameters works for any coin (penny, nickel or quarter) in any order.

3. Any coin, large or small, that subtends 1/5 rad in my field of view appears twice as tall as a coin that subtends 1/10 rad, and four times as tall as a coin that subtends 1/20 rad. Its actual size is taken into account by measuring with relative diameters rather than absolute centimeters.

4. Students will invent a broad range of challenges. See Lesson Note 4 at left.

**Test for Understanding**

Q: Will Jill see the relative sizes of these dimes the same as Jack? Draw a diagram to support your conclusion.

A: Both Jack and Jill see the nearer coin as larger (twice as large in this drawing).
Module C: COUNTING DIAMETERS

C5: Make a ruler from paper plates.

1. Tape a paper plate over dark background paper on your wall. Extend a Paper-Plate Ruler as follows:
   - a. Line up 25 paper plates on the floor. Alternate face-up and face-down.
   - b. Join them (up-rim touching down-rim) with masking tape.
   - c. Number the plates boldly from 1 to 25 from the wall.

2. Get your Coin Ruler.
   - a. Hold the zero end of the **nickel** scale facing up, directly under one eye.
      
      *(Pinch the scale from underneath to hold it in one hand.)*
   - b. Hold a nickel in your other hand.

3. Experiment by making the nickel “eclipse” the paper plate.
   - a. Compare nickel diameters on the Coin Ruler with plate diameters on the floor.
   - b. Report your findings in terms of subtended angles.

**Inquiry:**
**Materials:**
- black construction paper (optional), paper plates, masking tape, Coin Ruler, a nickel

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**Lesson Notes**

1. Black (or dark) construction paper makes a white paper plate more visible. If your wall is a dark color, this added contrast may not be needed.

   Generic 9-inch paper plates work best. They must be circular, without divided sections. You'll need to purchase 25 paper plates per ruler, but can economize by asking students to cooperatively make just a few rulers, then store them in a place available to all. Many students can use one ruler at the same time, so you can probably get by with just one or two rulers per class.

   1a. Alternating plates face-up and face-down allows the rulers to fold neatly for storage.

2. Notice that the Coin Ruler fits directly under the eye. Unlike using it on a table surface, no zero-point adjustment is required. Remind students to use care, of course, when placing objects close to their eyes.

3. Observant students may notice that the nickel doesn't quite cover the paper plate at close distances (large radian angles). What's going on? As you bring the nickel closer to your eye while focusing on the distant plate, the nickel becomes increasingly out of focus. As its edge blurs, the nickel looks smaller. But if you focus on the nickel (squinting helps), its increasingly distinct edge again covers the distant plate. A focused nickel continues to eclipse the plate at distances as close as 5 diameters. (See also lesson note 4b, page 75, lower left.)

   This experiment works, of course, with quarters and pennies, too. Students who find poor agreement between equal subtended angles and equal apparent size may be inadvertently confusing coin scales.

**Model Answers**

3a-b. When both nickel and paper plate are an equal number of their respective diameters from the eye, they subtend the same angle in my field of view, and therefore appear to have the same apparent size.

   For example, a nickel 17 diameters from the eye subtends $\frac{1}{17}$ rad in my field of view. Equally, the distant paper plate, 17 paper-plate diameters from where I stand, also subtends $\frac{1}{17}$ rad. (See Teaching Notes 3, this page.)

**Test For Understanding**

Q: Any object “n” diameters (or heights, or widths) from your eye subtends what angle in your field of view?

A: $\frac{1}{n}$ rads
C6: Does a Moon Ruler correctly scale the real thing?

1. Get half of a Moon Ruler page. Make a ruler that counts paper-punch diameters as follows:
   - Cut out the three numbered strips, and tape them in order. Use the edge of your desk as a guide.
   - Fold the ends as marked. Punch out the “moon” hole with a hole punch.

**modeling:** C1C2C3C4C5C6C7C8

**materials:** half of the Moon Ruler page, scissors, clear tape, hole punch

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2. Our moon (diameter = 3,480 km) orbits Earth (diameter = 12,740 km) at an average distance of 384,000 km. Show that your Moon Ruler accurately scales these distances:
   - Do the correct number of moons fit across Earth? Show your math.
   - Do the correct number of moons fit in between? Show your math.

3. What angular size is the moon as seen from Earth? How many degrees is this?

---

**MODEL ANSWERS**

2a. Earth diameter/moon diameter = 12,740 km / 3,480 km = 3.66.
   Thus \(\frac{3}{2}\) paper-punch “moons” span 1 “Earth,” as shown on the model.

2b. Earth-moon distance/moon diameter = 384,000 km / 3,480 km = 110.
   Thus 110 moon diameters fit between the Earth and moon as shown on the ruler.

3. Angular size of moon = \(\frac{1}{110}\) rad
   \(\frac{1}{110}\) rad x \(180^\circ/\pi\) rad = 0.52° = \(\frac{1}{2}\) degree

**TEST FOR UNDERSTANDING**

Q: How large would Earth appear if you observed it from the Moon? State your answer in radians and degrees.

A: 
   \(384,000 \text{ km} / 12,740 \text{ km} = 30\) Earths.

Also, 
   \(110\) moons x \(1\) Earth / \(3.66\) moons = 30 Earths.

Standing on the Moon, a “full Earth” would subtend \(\frac{1}{30}\) rad in your field of vision: \(\frac{1}{30}\) rad x \(180^\circ/\pi\) rad = 1.9°
Module C: COUNTING DIAMETERS

C7: Gaze at a paper plate moon.

1. Examine your Moon Ruler. Calculate how far from a paper plate you must stand so it looks as big as the real moon. Answer in both paper-plate diameters and in meters.

2. Cut a piece of string 10.0 meters long. Wrap it around a corrugated cardboard "spool."

3. Tape a plate to a tree or wall outside. Measuring with string and a meter stick, find your calculated distance.

4. View the plate through your Moon Ruler from that distance. What do you observe? (If your arms are short, crease the "moon" end of the ruler lengthwise to keep it straight.)

application: C1 C2 C3 C4 C5 C6 C7 C8

materials: Moon Ruler, scissors, a paper plate, a meter stick, string, masking tape, a piece of corrugated cardboard

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INTRODUCTION

Hold the "Earth" end of the Moon Ruler directly under your eye while you sweep the paper-punch "Moon" through a wide arc.

a. Must this tiny model "moon" be curved in order to match the curving circumference of the giant circle I'm tracing? No. Even though it's "flat," it's small enough to closely match the curving circumference of the much larger circle, with a radius equal to the viewing distance.

b. How many "moons" fit around this circumference? $2\pi \text{ rad} \times 110 \text{ paper punches} / 1 \text{ radius} = 691 \text{ paper punches}$

c. Knowing (b), how many degrees does a paper punch "moon" subtend? $360^\circ / 691 \text{ paper punches} = 0.52^\circ \approx 1/2 \text{ degree}$

d. How many radians does a "moon" subtend? $1/110 \text{ rad}$

e. How many degrees is this? $1/110 \text{ rad} \times 180^\circ / \pi = 0.52^\circ \approx 1/2 \text{ degree}$

LESSON NOTES

2. This string makes a useful measuring tool here and in many other Labs: D4, E4, E5, E6, F2, F4.

3. Students might stretch out the 10-meter string twice, then fold it in half to estimate the required 25 meters.

MODEL ANSWERS

1a. This answer is based on generic 9 inch (23 cm) paper plates. Other sizes work proportionally, of course, but won't work out to a standing distance of 25 meters.

In Lab C6, the angular diameter of the moon was shown to be $1/110$ rad. If the moon is represented by a paper plate, then I must stand 110 plates away to see it sized like the real moon in our night sky.

$(110 \text{ plates} / 1) \times (23 \text{ cm} / \text{ plate}) \times (1 \text{ m} / 100 \text{ cm}) = 25 \text{ m}$

3a. The paper-punch hole perfectly frames the paper plate at a distance of 25 meters.

TEST FOR UNDERSTANDING

This question anticipates Lab C8.

Q: Tape a shiny dime to a wall (or a window with horizon view for special effect). Calculate how far away you must stand so this dime has the same apparent size as a rising full moon. Assume $D_{DIME} = 1.785$ cm.

A: $(110 \text{ dimes} / 1) \times (1.785 \text{ cm} / \text{dime}) \times (1 \text{ m} / 100 \text{ cm}) = 1.96 \text{ m} \approx 2 \text{ meters}$
Module C: COUNTING DIAMETERS

C8: Test your Moon Ruler on family and friends.

1. Tape a shiny dime to a wall or window to represent the moon. Ask your family and friends to stand far enough away so this dime looks as big as the real moon.

2. Get the Dime Ruler page and assemble the ruler. Use it to measure each guess in radians from eye to dime.

3. First chance you get, show them the real moon outside, looking through the Moon Ruler.
   a. Write about your interactions.
   b. By what factors did they overestimate its size?

4. Are these results typical? If time allows, extend your research. Would you get the same results with a compact disk (CD) moon disk?

notes & vocabulary
Consult a calendar to find the current phase of the moon. You might spot it in the daytime sky if you know where to look:

A first quarter moon follows the sun across the sky: about 90° behind.

A last quarter moon leads the sun across the sky: about 90° ahead.

research: C1 C2 C3 C4 C5 C6 C7 C8
materials: dime, Dime Ruler page, Moon Ruler, clear tape, current calendar with moon phases

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LESSON NOTES
1. This is a wonderful opportunity for involving friends and family in an open-ended and fun science experiment.
2. Students should cut and tape this Dime Ruler, in the correct number sequence, using the edge of their table to keep the pieces straight. They'll need both this ruler and the Moon Ruler for home experimentation.

Radian angles are measured most accurately by using the Dime Ruler like a measuring tape, running it from the eye of the observer to the dime taped to the window or wall. Students can also frame the dime with their Moon Rulers to measure the angle in rough approximation.

3. The Moon Ruler perfectly frames the real moon, anywhere in the sky, in any phase. The effect is particularly dramatic with a full rising moon. It looks so large without the Moon Ruler, and yet it still fits the paper punch hole, just as it does when higher in the sky!

MODEL ANSWERS
1-2. Since a rising moon looms much larger, psychologically, than the angle it actually subtends in our field of view, most people will stand much too close to the dime, resulting in subtended angles much larger than 1/110 rad.

3. Reactions of disbelief and astonishment are common.

4. Lots of opportunity here to turn this activity into a science project!

TEST FOR UNDERSTANDING
Q: About 400 Moon diameters fit across 1 Sun diameter. What can you conclude about the relative distances of the Sun and Moon from Earth?
A: Since the Moon and Sun both have the same angular size, the Sun must be 400 times farther away.
Module D: VISUAL ACUITY

Astronomers use orbiting telescopes like GLAST and Hubble to view deep space objects with tiny angular sizes. These activities enable you to make a simple eye chart to test your own personal acuity, then compare your eyesight to these precise instruments.

MAP YOUR LABS: Complete D1 first. The remaining labs are up to you: D2 reinforces D1; D3 sharpens your math skills; D4 is an open-ended exploration you can do outside.

D1: Concept
Test your visual acuity with a 1 mm dot.
How many millimeters back from a 1 mm dot can you stand and still make it out? This distance divided into 1 mm is your visual acuity in radians.
(Time: 30 minutes)

D2: Reinforcement
Test your visual acuity with other dot diameters.
What works for a 1 mm dot works for smaller and larger test dots too. Express radians as fractions with 1 in the numerator for maximum clarity!
(Time: 50 minutes)

D3: Application
Compare your eyesight with GLAST and Hubble.
Practice converting user friendly radians back into seconds, minutes and degrees.
(Time: 30 minutes)

D4: Inquiry
Can you see a dime at 100 meters?
Make a prediction, plan your experimental strategy and report your conclusions.
(Time: 40 minutes)
D1: Test your visual acuity with a 1 mm dot.

1. Get the mm Dots page. Cut out only the grey-bordered square.
2. Ask a partner to hold this 1-mm dot (d) against a wall while you view it from a distance of 1,000 mm (D). What angle d/D does this 1 mm dot subtend in your visual field?
3. Measure how far away you can stand (to the nearest 100 mm), and still see the dot:
   a. To prevent "imagining" dots, ask your lab partner to hold the square in different "diamond" positions over multiple trials, while you identify the correct dot quadrant.
   b. Report the smallest radian angle you can see without error. This is your visual acuity.

4. Retest your acuity when the one-mm dot is illuminated by direct sunlight. (Keep your back to the sun.)
   a. Report the smallest radian angle that you can see without error.
   b. How does lighting affect your visual acuity? Generalize.

concept: D1 D2 D3 D4
materials: Dots page, scissors, a meter stick, a wall under constant lighting

D2: Test your visual acuity with other dot diameters.

1. Copy this data table on notebook paper.
   a. On line (3) under 1.0 mm Dot, fill in your personal experimental MAX distance determined in Lab D1.
   b. Cut out the remaining mm Dot squares in this table.

2. Predict MAXimum distances (to the nearest 100 mm), that you will be able to stand from each dot and still see it:
   a. Show your math. Explain your reasoning.
   b. Complete line (2) of your data table.

3. Work with a partner using the same wall and the same lighting as Lab D1. Fill in line (3), reporting distances to the nearest 100 mm.

4. Compute d/D using experimental (not predicted) distances. Convert fractions so "1" is in the numerator.
   a. Fill in line (4). Account for differences in these fractions.
   b. Fill in line (5). This might be an averaged fraction, a rounded fraction, whatever you think your visual acuity really is.
Module D: VISUAL ACUITY

D3: Compare your eyesight with GLAST and Hubble.

1. Use these conversion factors and a calculator to find precise values for the last four approximated table entries.

<table>
<thead>
<tr>
<th>DEGREES</th>
<th>RADIANS</th>
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</thead>
<tbody>
<tr>
<td>180°</td>
<td>π</td>
</tr>
<tr>
<td>57.3°</td>
<td>1</td>
</tr>
<tr>
<td>1°</td>
<td>1/57.3</td>
</tr>
<tr>
<td>(1/2)°</td>
<td>1/114.6</td>
</tr>
<tr>
<td>1 arc minute</td>
<td>1/3,400 (approx)</td>
</tr>
<tr>
<td>30 arc seconds</td>
<td>1/7,000 (approx)</td>
</tr>
<tr>
<td>1 arc second</td>
<td>1/200,000 (approx)</td>
</tr>
<tr>
<td>1/5 arc second</td>
<td>1/1,000,000 (approx)</td>
</tr>
</tbody>
</table>

2. Express your maximum visual acuity in radians. In arc seconds.

3. GLAST can detect gamma-ray sources as small as 30 arc seconds, while the Hubble Space Telescope can detect visible light sources down to 0.1 arc second.
   a. At what maximum distance can GLAST see a 1 mm gamma-ray source? Can Hubble see a 1 mm light source? Can you see a 1 mm light source?
   b. How many times better or worse is your eyesight than GLAST's? Than Hubble's?

application: D1 D2 D3 D4
materials: Calculator
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D4: Can you see a dime at 100 meters?

1. Predict.

2. Experiment.
   a. Does background matter? Would a background sheet divided into quadrants be useful? What about color and size?
   b. How will you communicate over long distances?
   c. How will you accurately measure your maximum viewing distance?

3. Conclude.

I wonder what he's trying to tell me...

That's too far!

mm ruler

| 50 | 50 |
| 40 | 40 |
| 30 | 30 |
| 20 | 20 |
| 10 | 10 |
| 0  | 0  |

inquiry: D1 D2 D3 D4
materials: a dime, masking tape, black nonreflective paper, suitable open space, measuring string, calculator
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page 51
mm Dots

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**D1: Test your visual acuity with a 1 mm dot.**

1. Get the mm Dots page. Cut out only the grey-bordered square.
2. Ask a partner to hold this 1-mm dot (d) against a wall while you view it from a distance of 1,000 mm (D). What angle d/D does this 1 mm dot subtend in your visual field?
3. Measure how far away you can stand (to the nearest 100 mm), and still see the dot:
   a. To prevent “imagining” dots, ask your lab partner to hold the square in different “diamond” positions over multiple trials, while you identify the correct dot quadrant.
   b. Report the smallest radian angle you can see without error. This is your visual acuity.

4. Retest your acuity when the one-mm dot is illuminated by direct sunlight. (Keep your back to the sun.)
   a. Report the smallest radian angle that you can see without error.
   b. How does lighting affect your visual acuity? Generalize.

---

**INTRODUCTION**

Tightly wad a piece of 8½ x 11 inch scratch paper and show it to your class. Challenge your class to tell you what angle it subtends in their field of vision:

a. What do we need to know? The diameter of the ball (d) and its distance to you the observer (D).

b. How will we measure? Hold up a meter stick to show that the paper wad measures about 5 cm in diameter, or 20 wads per meter stick. Then estimate the distance to Denzel's chair, in end-to-end meter sticks (20 wads, 40 wads, 60 wads, ..., 120 wads).

c. How large does this paper wad appear to Denzel?

\[
\theta = \frac{d}{D} = \frac{1}{120} \text{ rad}
\]

This paper wad is \(1\over120\) of the radius of a large circle with Denzel’s eye at the center.

**LESSON NOTES**

2. Designate a testing wall with these characteristics: (a) Constant, even illumination. Today’s brightness will be the same as tomorrow’s. Ordinary electric lighting is ideal; natural lighting may be too variable. (b) Up to 12 meters of free approach. Less distance is OK, but will limit the data table in D2 to fewer entries.

This d/D terminology communicates important ideas. “Little d” always stands for the diameter (or height or width) of the object you are looking at. And “big D” always represents the radial distance between you and that object, expressed in the same units as “little d.” The fraction d/D is the apparent angular size (in radians) of what you see, as long as the angles remain small. For this reason, d/D is often called the small angle approximation.

3. Each student takes a turn at being both “testee” and “tester.” The tester should rotate the dot to new positions to be sure the testee isn’t just making lucky guesses. Students will naturally compete, so require sensitivity toward the needs of those who are visually impaired.

**MODEL ANSWERS**

2. subtended angle = d/D = 1 mm/1,000 mm = 1/1,000 rad.
3b. Expect results between 1/3,000 and 1/6,000 rad, depending on how well your testing wall is illuminated.

4a. Expect results between 1/6,000 and 1/8,000 rad in direct sunlight.

4b. Lighting dramatically affects visual acuity. In brighter light, acuity increases; that is, you can see smaller objects. This is not surprising. In total darkness, a dot is simply not visible at any distance. However, intense, glaring sunlight also makes the dot harder to see.

**TEST FOR UNDERSTANDING**

Q: Somebody tosses a 1 meter beach ball in the air at a distance of 1 kilometer in broad daylight. Can you see it?
A: Yes. d/D = 1 m/1,000 m = 1/1,000 rad. My visual acuity is sharp enough to see angles several times smaller than this apparent size.
Module D: VISUAL ACUITY

D2: Test your visual acuity with other dot diameters.

1. Copy this data table on notebook paper.
   a. On line (3) under 1.0 mm Dot, fill in your personal experimental MAX distance determined in Lab D1.
   b. Cut out the remaining mm Dot squares in this table.

   (1) dot diameters (d) in mm: 0.5 1.0 1.5 2.0
   (2) predicted MAX distance (D) in mm: 3000 9000 12000
   (3) experimental MAX distance (D) in mm: 2900 6000 8800 ***
   (4) experimental MIN angular size (d/D) in rad: 1/5800 1/6000 1/5867 ***
   (5) accepted acuity (d/D) in rad: 1/5900

   reinforcement: D1 D2 D3 D3

   materials: Dots page, scissors, a meter stick, a calculator, acuity results from D1.

2. Predict MAXimum distances (to the nearest 100 mm), that you will be able to stand from each dot and still see it:
   a. Show your math. Explain your reasoning.
   b. Complete line (2) of your data table.

3. Work with a partner using the same wall and the same lighting as Lab D1. Fill in line (3), reporting distances to the nearest 100 mm.

4. Compute d/D using experimental (not predicted) distances. Convert fractions so “1” is in the numerator.
   a. Fill in line (4). Account for differences in these fractions.
   b. Fill in line (5). This might be an averaged fraction, a rounded fraction, whatever you think your visual acuity really is.

LESSON NOTES
1-5. These steps are structured so students will predict, experiment and evaluate, in that order. Determining maximum dot-viewing distances by trial and error experimentation, before reaching for a meter stick, insures a range of distances that may not agree with predicted values. This offers students real experience with the messy world of experimental uncertainty.

MODEL ANSWERS
1-5. A typical table might contain data like this:

2. In Lab D1, the 1.0 mm Dot was visible at a MAX distance of 6000 mm, resulting in a MIN subtended angle of 1/6000 rad. Assuming that my eyes will see the other dots with the same acuity...
   0.5 mm dot: Half this dot diameter will be visible up to half this distance (6,000 mm / 2 = 3,000 mm).
   1.5 mm dot: 1.5 times this dot diameter will be visible up to 1.5 times this distance (6,000 mm x 1.5 = 9,000 mm).
   2.0 mm dot: Twice this dot diameter will be visible up to twice this distance (6,000 mm x 2 = 12,000 mm).

4. 0.5/2,900 = 1/5,800; 1.5/8,800 = 1/5,667
4a. This range of distance uncertainty (200 mm or 20 cm) indicates that you can stand up to 10 cm closer or farther away from the dots with no noticeable changes in your ability to see the dots.

TEST FOR UNDERSTANDING
Q: GLAST is equipped with solar panels that span 14.7 meters from tip to tip. And it orbits Earth at an altitude of 550 km.
   a: How many of these spans fit between you and GLAST as it orbits directly overhead?
   b: Can you see it from Earth?

A: a. 14.7m/550,000 m = 1 span/37,000 spans
   b. GLAST subtends 1/37,000 rad in my field of view, exceeding my visual acuity by more than a factor of 6. It may be visible to the naked eye, however, in a dark sky as it reflects sunlight.
D3: Compare your eyesight with GLAST and Hubble.

1. Use these conversion factors and a calculator to find precise values for the last four approximated table entries.

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2. Express your maximum visual acuity in radians. In arc seconds.

3. GLAST can detect gamma-ray sources as small as 30 arc seconds, while the Hubble Space Telescope can detect visible light sources down to 0.1 arc second.

   a. At what maximum distance can GLAST see a 1 mm gamma-ray source? Can Hubble see a 1 mm light source? Can you see a 1 mm light source?
   
   b. How many times better or worse is your eyesight than GLAST's? Than Hubble's?

Application: D1 D2 D3 D4

Materials: Calculator

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MODEL ANSWERS

1. Students should compare each precise calculation against the approximate values listed in the table:

   (1 arc minute/1) x (1°/60 arc minutes) x (π rad/180°) = 0.0002909 rad = 1/3,438 rad

   30 arc seconds = 1/2 x 1/3,438 rad = 1/6,876 rad

   (1 arc second/1) x (1 arc min/60 arc seconds) x (1°/60 arc min) x (π rad/180°) = 4.848 x 10^-6 rad = 1/206,280 rad

   0.2 arc second = 1/5 x 1/206,280 rad = 1/1,031,400 rad

2. This calculation, based on a visual acuity of 1/8,000 rad in direct sunlight, fits below 30 arc seconds in the table.

   1/8,000 rad x 180°/π rad x 60 arc minutes/1° x 60 arc seconds/1 arc minute = 26 arc seconds

3a. GLAST:

   30 arc seconds = 1/7,000 rad.
   It can see a 1 mm gamma-ray source up to about 7,000 mm, or 7 meters away.

   Hubble:

   0.1 arc second = 1/2,000,000 rad.
   It can see a 1 mm visual source up to 2,000,000 mm, or 2 kilometers away.

   Me:

   1/8,000 rad. I can see a 1 mm visual source up to 8 meters away.

3b. GLAST, in the gamma-ray spectrum, has about the same acuity as I do in the visible spectrum.

   Hubble, in the visible spectrum, can see much, much better than I can: 2,000,000 mm/8,000 mm = 250 times better.

TEST FOR UNDERSTANDING

Q: The Moon has an elliptical orbit around the Earth. At its farthest (apogee), it subtends 1/117 rads as observed from Earth. At its nearest (perigee), it subtends 1/103 rads. Calculate its angular variation in rads and degrees.

   A: Variation = 0.00971 rad – 0.00855 rad
                 = 0.00116 rad

   (0.00116 rad/1) x (180°/π rad) x (60 arc min/1°) = 3.98 arc min

   Variation = 3.98 arc min

   Variation = 3', 58"

Page 55
D4: Can you see a dime at 100 meters?

1. Predict.
2. Experiment.
   a. Does background matter? Would a background sheet divided into quadrants be useful? What about color and size?
   b. How will you communicate over long distances?
   c. How will you accurately measure your maximum viewing distance?
3. Conclude.

MODEL ANSWERS

1. Predict:
   The diameter of a dime is 18 mm. Students can make this estimate by placing the dime directly on the printed ruler.

   Students have previously estimated their visual acuity to range somewhere between 1/4,000 rad in ordinary room light to 1/8,000 rad in direct sunlight. On a bright cloudy day, they might assume it to be somewhere in the middle, say, 1/6,000 rad.

   So how far away can a dime be, before it gets too small to see? Applying the small angle formula:

   \[ \frac{d}{D} = \frac{18 \text{ mm}}{D \text{ mm}} = \frac{1}{6,000} \]

   \[ D = 18 \text{ mm} \times 6,000 = 108,000 \text{ mm} = 108 \text{ meters} \]

2. Experiment:
   a. The mm test dots were black on a square with white background. To provide similar contrast, the silver-colored dime would best stand out against a dark nonreflective background like black construction paper. Cut the paper square and center the dime in one of 4 quadrants, so the paper can be turned during testing, as before.
   b. Communication between the tester and testee must happen over long distances. Shouting might work if that won’t disturb the neighborhood, or a system of hand signals might be agreed upon in advance of experimentation.
   c. A premeasured 10 meter string makes a convenient ruler. Balls of scrap paper might be used to mark 10 m increments along the way so students don’t lose count.

3. Conclude:
   Under cloudy conditions, most students will be able to clearly see a dime at 100 meters, perhaps much farther if viewing it in direct sunlight (back to the sun). Or, using the dime like a tiny mirror, it may be possible to see flashes of reflected light over much longer distances.

TEST FOR UNDERSTANDING

Q: If an average car or truck is 6 feet wide, can you see traffic from an airplane window cruising at an altitude of 36,000 feet?

A: Yes, \( \frac{d}{D} = \frac{6 \text{ ft}}{36,000 \text{ ft}} = \frac{1}{6,000} \text{ rad} \)

Objects that subtend an angle of this size may be possible to see given suitable conditions: sunny clear weather, a clean airplane window, a well-positioned freeway to study.
Module E: TIE YOUR KAMAL

Ancient Arabian navigators invented the Kamal, a nifty tool for estimating subtended angles. A string held the Kamal a constant distance (D) from the sailor’s eye. Our modern adaptation replaces a traditional wooden rectangle with an adjustable paper triangle that folds to any width (d). Match its subtended angle (d/D) to any object beyond your nose – to people, cars, distant trees or buildings, a model Sun, the real Moon, anything that’s not too close. Instantly you can measure its apparent angular size in your field of view and estimate its distance from your eye!

**MAP YOUR LABS:** Make a Kamal (Lab E1) to start. All remaining labs require this tool, except E7 which is a stand-alone research project. You may do E2 or E3 because they are parallel activities, but it’s best to do E4 and E5 because they build on each other. Lab E6 is a very cool astronomy application you’ll do outside.

**E2: Practice** Learn to ride your Kamal.
Practicing good posture, notice where to stand along a row of paper plates so Kamal and plate radial angles correspond.
(Time: 30 minutes)

**E1: Toolmaking** Make a paper Kamal; try it out.
Merge 1/10 radian (5.73°) and a meter stick at a distance of 10 meters. Consider the implications!
(Time: 30 minutes)

**E3: Practice** What’s your apparent shoe size?
Practice estimating apparent angular size using the edge calibrations on your Kamal. Improvise a Kamal with your thumb and outstretched arm.
(Time: 30 minutes)

**E4: Application** Knowing HEIGHT, estimate distance.
Do this with a lab partner. Estimate her distance from you when she appears 1/10 radian tall.
(Time: 30 minutes)

**E5: Application** Knowing DISTANCE, estimate height.
Work with a lab partner as before. Estimate his height with your Kamal at a distance of 30 meters.
(Time: 20 minutes)

**E6: Modeling** Lay out the inner solar system plus Jupiter.
Space the planets with your Kamal so the Sun has the correct apparent size when viewed from each planet.
(Time: 1.5 hours)

**E7: Research** Do other planets have larger moon-rises?
Collect moon data from the library or on the net. Calculate the ratio of diameter to distance for all major moons in our solar system.
(Time: 1.5 hours)
E1: Make a paper Kamal and try it out.

1. Get the Paper Kamal page.
   a. Cut out one of the triangles. Be accurate.
   b. Fold in half across the $\frac{1}{16}$ radian line.

2. Continue folding across all full lines up and down the triangle:
   a. Take time to make neat, symmetrical folds directly on each line. A thin straightedge may be helpful.
   b. Don't fold at the shorter marks along the edges at this time.

3. Cut 55 cm of string. Tie a knot at one end.
   a. Fold a "flag" of tape over the other end to leave precisely 50 cm of string to the knot.
   b. Write your initials on the tape and the back of the Kamal.

4. To use your Kamal, fold it at $\frac{1}{10}$ radian, and hold the tape flag just below the fold.
   a. Hold the knot at your eye (or in your teeth), and extend your Kamal to the end of the string. The folded edge now subtends $\frac{1}{10}$ radian in your visual field.
   b. Confirm this visual angle by taping a meter stick to the wall and standing 10 meters away. Explain what you observe.

E2: Learn to ride your Kamal.

1. Tape a paper plate to your wall. String 25 numbered paper plates underneath, like a ruler. (See Lab C5.)

2. Practice "riding your Kamal" up and down this ruler: Where should you stand along the Paper-Plate Ruler to accurately frame the wall plate with your Kamal folded at these angles?
   a. $\frac{\pi}{9}$ rad, $\frac{1}{22}$ rad, $\frac{1}{12.5}$ rad
   b. 0.1 rad, 0.05 rad, 0.01 rad
   c. 6°, 3°, 1°
   d. $\frac{\pi}{90}$ rad, ($\frac{1}{2}$)°, 0.001 rad

3. Summarize how to "ride your Kamal":
   a. Do you prefer to "tie your Kamal" directly under your eye, or between your teeth? Why?
   b. With your back straight and chin down, what part of your feet align with the Paper-Plate Ruler to most accurately and consistently frame the wall-mounted plate: Toes? Arches? Ankles? Heels?
E3: What's your apparent shoe size?

1. Pace 14 shoe lengths (heel to toe) from a wall.

   a. Remove one shoe to mark this distance. Remove the other to lean upright against the wall.

   b. From the marker shoe, what is the apparent angular size of the upright shoe? Confirm your prediction with your Kamal.

2. Repeat this experiment for a 3° subtended angle.

3. Determine where to place your feet (relative to your "marker") to accurately frame your wall shoe every time. Where is this for you?

4. What other kinds of objects, besides shoes, could you measure with?

5. How many thumb widths fit between your eye and outstretched thumb? How did you figure this out?

   This many shoe-lengths back, the thickness of my thumb now matches the length of my shoe.

practice: E1 E2 E3 E4 E5 E6 E7

materials: your shoes, a Kamal

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E4: Knowing HEIGHT, estimate distance.

1. Work with a lab partner. Measure each other's height against a wall with a meter stick. (Protect the wall with masking tape.)

2. Imagine that your lab partner walks away from you until her apparent height is precisely 1/10 radian as measured by your Kamal.

   Stop there! That's 1/10 radian.

   10 BODY LENGTHS

   a. Calculate her distance from you in meters when she appears 1/10 radian tall.

   b. Do this experiment in a hallway, gym or outside. Measure the actual distance that separates you with a meter stick and/or a 10-meter string.

3. Calculate your experimental error: \[ \% \text{error} = \left( \frac{\text{difference}}{\text{actual distance}} \right) \times 100 \]

   application: E1 E2 E3 E4 E5 E6 E7

   materials: a meter stick, masking tape, Kamal, 10-meter string (optional, see lab C7), a calculator

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4. Did you get the same distance between as your lab partner did? Why?
E5: Knowing DISTANCE, estimate height.

1. Work in a hallway, gym or outside with the same lab partner as in activity E4.
   a. Stand 30 meters apart.
   b. Accurately measure and record each other’s apparent height in radians using your Kamals.

2. Calculate your lab partner’s actual height in meters. Write a proportion:
   \[ \frac{1}{x \cdot x \text{ rad}} = \frac{d_{\text{partner's height}}}{D_{\text{distance to partner}}} \]

3. How close did you come to the actual height of your lab partner measured in Lab E4.

   \[ \% \text{ error} = \left( \frac{\text{difference}}{\text{actual height}} \right) \times 100 \]

application: E1 E2 E3 E4 E5 E6 E7
materials: a meter stick, masking tape, Kamal, 10-meter string (optional, see lab C7), a calculator

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E6: Lay out the inner solar system plus Jupiter.

1. Get the Planets to Scale page.
   a. Calculate (on notebook paper), the angular size of the Sun as observed from each planet in \( \frac{d}{D} \) rad. Simplify to \( 1/n \) rad, then record each result in the box for that planet.

   Sun’s apparent size from this planet:

   Tape here

b. Use a hand lens to read the smallest calibrations on your Kamal. What do you notice?

2. Cut out the “Sun” patch. Tape it to the back of a paper plate so its curved edge meets the rim.
   a. Measure the diameter of this paper-plate Sun to the nearest 0.01 meters. Record it in the box provided.
   b. Cut into the plate along each side of this patch. Slip the flap into a small jar so the “Sun” stands upright. (Tape in place if needed.)

3. Cut and tape the planet patches. “Jupiter” may require extra tape so it doesn’t flop over.

4. Gather these items in a box or bag to take outside: model planets and Sun, Kamal, 10-meter string, lab instructions, paper and pencil, calculator, clip board (optional).

5. Lay out this model Solar System along a sidewalk, road, track or field. Use your Kamal to place each planet far enough back from the “Sun” to match the apparent size you calculated.

6. Use your 10-meter string to evaluate how accurately you placed Earth. (Check out other planets if you have time.)

7. Turn the planets so you can see each one from “Earth.” Compare their visibility (apparent sizes) to what you see in the night sky.

modeling: E1 E2 E3 E4 E5 E6 E7
materials: Planets to Scale page, calculator, hand lens, nine-inch paper plate, masking tape, scissors, 6 small jars or cans, Kamal, grocery bag or box, 10-meter string, 130 meters of clear space (a sidewalk or field), clipboard (optional)

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E7: Do other planets have larger moon-rises?

Is there any planet in our solar system that has a rising moon that looks apparently larger than our own Luna?
(Assume clear views, unimpeded by atmosphere or darkness.)

1. What important variables must you research?
2. Report your results.

research: E1 E2 E3 E4 E5 E6 E7
materials: calculator, research materials from a library or internet access.

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Module E: TIE YOUR KAMAL

Kamal Triangle

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Planets To Scale

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<table>
<thead>
<tr>
<th>Planet</th>
<th>Diameter</th>
<th>To Sun</th>
<th>Sun's apparent size from this planet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>4,878 km</td>
<td>57,900,000 km</td>
<td></td>
</tr>
<tr>
<td>Venus</td>
<td>12,104 km</td>
<td>108,000,000 km</td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>12,756 km</td>
<td>150,000,000 km</td>
<td></td>
</tr>
<tr>
<td>Mars</td>
<td>6,794 km</td>
<td>228,000,000 km</td>
<td></td>
</tr>
</tbody>
</table>

Sun

(A Paper Plate)

Diameter: 1,390,000 km

Paper plate diameter: (meters)

Jupiter

Diameter: 138,150 km

Sun's apparent size from this planet:
Module E: TIE YOUR KAMAL

E1: Make a paper Kamal and try it out.

1. Get the Paper Kamal page.
   a. Cut out one of the triangles. Be accurate.
   b. Fold in half across the 1/16 radian line.

2. Continue folding across all full lines up and down the triangle:
   a. Take time to make neat, symmetrical folds directly on each line. A thin straightedge may be helpful.
   b. Don’t fold at the shorter marks along the edges at this time.

3. Cut 55 cm of string. Tie a knot at one end.
   a. Fold a “flag” of tape over the other end to leave precisely 53 cm of string to the knot.
   b. Write your initials on the tape and the back of the Kamal.

4. To use your Kamal, fold it at 1/10 radian, and hold the tape flag just below the fold:
   a. Hold the knot at your eye (or in your teeth), and extend your Kamal to the end of the string. The folded edge now subtends 1/10 radian in your visual field.
   b. Confirm this visual angle by taping a meter stick to the wall and standing 10 meters away. Explain what you observe.

Introduction

A Kamal is a simple navigational device invented by the ancient Arabs. It was a length of knotted string tied to a bar of wood. By holding knots in the string between his teeth (each knot set to the latitude of a specific port), a sailor could extend the wood Kamal between his horizon and Polaris, the North Star. If the wood didn’t span the distance precisely, he would sail south until Polaris’ altitude “lowered” to meet his Kamal, or north to make Polaris “rise.” When the wood spanned the gap precisely, he could then “sail his latitude” east or west to reach his chosen port.

Prepare a Kamal in advance to show to your class. Like traditional Kamals, this TOPS Kamal can also be held between the teeth, but at only one position (50 cm from the eye). Unlike a traditional bar of wood, it uses a folded triangle to measure variable angles in radians and degrees.

Lesson Notes

2a. The triangle must be folded almost 40 times, across each solid line labeled in radians and each dotted line labeled in degrees. It takes perhaps 5 minutes to do a good job. This is time well spent. There are many choice experiments to do with a completed Kamal.

A good way to achieve accuracy is to maintain left-right symmetry as you complete each horizontal fold. Folding each line over a thin straightedge, or pressing the line against the edge with a finger to initiate each fold, may make for neater results.

3b. The Kamal has two separate parts: the calibrated triangle and the string. Both parts should be initialed, especially the string, which might be held in the mouth.

4. Pinch the tape tab against the Kamal, far enough below the 1/10 radian line to provide an unobstructed view across the folded edge. (Some students might paper-clip the tab to the Kamal until they can manage separate parts.) When the string is pulled straight, you have a 5 cm folded edge held 50 cm from the eye, which defines 5/60 (or 1/10) rad.

The folded edge is the “working” part of the Kamal. It can be held in any orientation (horizontally, vertically, or diagonally), and folded at variable lengths to visually match the apparent angular size of any object in d/D radians.

Model Answers

5b. The Kamal’s folded edge, labeled 1/10 radian, perfectly matches the length of the meterstick at 10 meters. The Kamal confirms that 1 meter, viewed from a distance of 10 meters, subtends 1/10 radian in my field of view.

Test for Understanding

If you stood 8 meters from a meter stick, where must you fold your Kamal to match the meter stick’s apparent height? Fold it along the top line, labeled 1/8 radian.
E2: Learn to ride your Kamal.

1. Tape a paper plate to your wall. String 25 numbered paper plates underneath, like a ruler. (See Lab C5.)

2. Practice “riding your Kamal” up and down this ruler: Where should you stand along the Paper-Plate Ruler to accurately frame the wall plate with your Kamal folded at these angles?
   - a. \( \frac{1}{9} \) rad, \( \frac{1}{22} \) rad, \( \frac{1}{12.5} \) rad
   - b. 0.1 rad, 0.05 rad, 0.01 rad
   - c. 6°, 3°, 1°
   - d. \( \frac{\pi}{90} \) rad, \( \left( \frac{1}{2} \right)^\circ \), 0.001 rad

3. Summarize how to “ride your Kamal”:
   - a. Do you prefer to “tie your Kamal” directly under your eye, or between your teeth? Why?
   - b. With your back straight and chin down, what part of your feet align with the Paper-Plate Ruler to most accurately and consistently frame the wall-mounted plate: Toes? Arches? Ankles? Heels?

practice: E1 E2 E3 E4 E5 E6 E7
materials: paper plates, masking tape, Kamal, hand lens (optional)

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INTRODUCTION

Notice radian angles on the Kamal are subdivided along each side. Take a “walk” with your class down these calibrations, starting at the top of the triangle and working toward the bottom point: \( \frac{1}{8.0}, \frac{1}{8.1}, \ldots, \frac{1}{15}, \frac{1}{15.2}, \ldots, \frac{1}{19}, 3^\circ, \frac{1}{19.2}, \ldots, \frac{1}{21}, \frac{1}{21.5}, \ldots, \frac{1}{32}, \frac{1}{33}, \ldots, \frac{1}{70}, \frac{1}{75}, \ldots, \frac{1}{150}, \frac{1}{164}, \frac{1}{559}, \frac{1}{1000}. \)

Radian angles \( \frac{1}{164} \) and \( \frac{1}{559} \) are useful in Lab E6. The smallest unlabeled mark at the very bottom of the triangle is calibrated to \( 1/1000 \) rad.

MODEL ANSWERS

2. Stand these numbers of paper-plates back from the wall.
   - a. \( \frac{1}{9} \) rad @ 9 plates
   - b. \( \frac{1}{22} \) rad @ 22 plates
   - c. \( \frac{1}{12.5} \) rad @ 12.5 plates
   - d. 6° @ 9.55 plates
   - e. 3° @ 19.1 plates
   - f. 1° @ 57.3 plates
   - g. \( \frac{\pi}{90} \) rad @ 28.7 plates
   - h. \( \left( \frac{1}{2} \right)^\circ \) rad @ 115 plates
   - i. 0.001 rad @ 1000 plates

   lowest tiny (unlabeled) hatchmark on the skinny apex

   \( \left( \frac{1}{2} \right)^\circ \) subtends 28.7 plates x 4

   \( \frac{\pi}{90} \) rad = \( 2^\circ \) subtends 28.7 plates

3a. Holding the Kamal under the eye, or in the traditional way between the teeth, is a matter of personal preference (though the teeth leaves one hand free). For good hygiene, students should NOT trade Kamal strings.

3b. Answers will vary according to individual differences. The idea is to find a consistent place to stand among the paper plates that gives reliable agreement between the Paper Plate standard and the Kamal. For different body types and vision capabilities, this may be toes or heels or someplace in between. In a sense, this activity “calibrates” the Kamal to body shape, posture, eyesight and personal preferences.

TEST FOR UNDERSTANDING

Q1: How will you hold your body to make accurate Kamal measurements?

Back straight, chin down.

Q2: A strip of masking tape accurately defines some distance to an object you wish to measure with your Kamal. Where will you place your feet in relation to this tape?

Students should specify what part of their foot aligns with the masking tape for consistent, accurate Kamal readings.
E3: What's your apparent shoe size?

1. Pace 14 shoe lengths (heel to toe) from a wall.
   
   a. Remove one shoe to mark this distance. Remove the other to lean upright against the wall.
   
   b. From the marker shoe, what is the apparent angular size of the upright shoe? Confirm your prediction with your Kamal.
   
   2. Repeat this experiment for a 3° subtended angle.
   
   3. Determine where to place your feet (relative to your “marker”) to accurately frame your wall shoe every time. Where is this for you?
   
   4. What other kinds of objects, besides shoes, could you measure with?
   
   5. How many thumb widths fit between your eye and outstretched thumb? How did you figure this out?

   This many shoe-lengths back, the thickness of my thumb now matches the length of my shoe.

practice: E1 E2 E3 E4 E5 E6 E7

materials: your shoes, a Kamal

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LESSON NOTES

1. Students previously calibrated their Kamals against a Paper Plate Ruler. This lesson reinforces the last, using shoe lengths as a calibration standard instead of paper plates. Again, the idea is to learn where to stand (with good posture), relative to the baseline shoe to get consistent and accurate Kamal readings. Students who experience difficulty getting in and out of their shoes might draw their shoe prints on cardboard templates.

MODEL ANSWERS

1b, 2. The subtended angle for a shoe at 14 shoe lengths is 1/14 rad. (Since 1/14 is not a labeled division, students will have to read the side calibrations and make a new fold at that level (above 4°).

   The subtended angle for a shoe viewed from 19.1 shoe lengths is 3°. Again, students must fold their Kamals at this new calibration to confirm the 3° angle.

3. Students should report results that are consistent for how they use their Kamals (between the teeth or under the eye), and how they hold their bodies. For some this will be the middle of the foot, for others the toes or the heel. Those who hold the paper triangle precisely 50.0 cm from their eye will find themselves standing, relative to their marker shoe, where a plumb line dropped from their eye meets that part of the foot.

4. Just about any object might work. Apparent body lengths could be measured outside on a dry, grassy field. Apparent moon diameters could be measured out the window of a space craft returning from the moon to Earth.

5. METHOD ONE:
   Count how many heel-toe paces (with shoes on!) you must walk until the width of your thumb, held at arm’s length, exactly blocks your shoe propped upright against the wall. If you counted 25 paces, then your thumb subtends 1/25 radian. (This angle could be confirmed with a Kamal.)

   METHOD TWO:
   Use a meter stick to measure the width of your thumb and how far you hold it in front of your eye with your arm held straight. Here is one calculation:

   \[
   \frac{d_{\text{thumb width}}}{D_{\text{thumb distance}}} = \frac{2.5 \text{ cm}}{65 \text{ cm}}
   \]

   \[
   d/D = 2.5 \text{ m} / 65 \text{ m} = 0.03846 \text{ rad} = \frac{1}{25} \text{ rad}
   \]

TEST FOR UNDERSTANDING

Q: The width of your outstretched thumb just eclipses the diameter of your friend’s head. What can you conclude about these objects?

A: Your thumb and your friend’s head both subtend equal angles to your eye. Also, the number of thumb widths from the end of your arm to your eye equals the number of head widths from where your friend is standing to your eye.
E4: Knowing HEIGHT, estimate distance.

1. Work with a lab partner. Measure each other's height against a wall with a meter stick. (Protect the wall with masking tape.)

2. Imagine that your lab partner walks away from you until her apparent height is precisely \( \frac{1}{10} \) radian as measured by your Kamal.

   \[ \text{Stop there! That's } \frac{1}{10} \text{ radian.} \]

3. Calculate your distance from you in meters when she appears \( \frac{1}{10} \) radian tall.

4. Do this experiment in a hallway, gym or outside. Measure the actual distance that separates you with a meter stick and/or a 10-meter string.

   \[ 10 \text{ BODY LENGTHS} \]

   \[ \text{a. Calculate her distance from you in meters when she appears } \frac{1}{10} \text{ radian tall.} \]

   \[ \text{b. Do this experiment in a hallway, gym or outside. Measure the actual distance that separates you with a meter stick and/or a 10-meter string.} \]

   \[ 3. \text{ Calculate your experimental error: } \% \text{ error} = \left( \frac{\text{difference}}{\text{actual distance}} \right) \times 100 \]

**Application:** E1 E2 E3 E4 E5 E6 E7

**Materials:** a meter stick, masking tape, Kamal, 10-meter string (optional), see lab C7, a calculator

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** LESSON NOTES **

1. Snippets of masking tape cut into triangles can be used as markers for measuring height, or students might fix masking tape to the wall and then mark on the tape itself with ink or pencil. The tape should be removed within a day or so, or it could leave sticky residue behind.

2. Any apparent size smaller than \( \frac{1}{6} \) radian also works. We used \( \frac{1}{10} \) radian to simplify the math. Much longer distances are theoretically possible to measure, though not very accurate. Someone 5 feet 10 inches tall, for example, would subtend \( \frac{1}{580} \) radians in your field of view when standing 1 kilometer away. (Find this smallest labeled calibration on the extreme tip of the Kamal just inside the unlabeled \( \frac{1}{1000} \) rad mark.)

**MODEL ANSWERS**

Answers will vary. Here is one set of calculations:

1. Actual height of lab partner = 1.72 meters
2a. 10 body lengths = 17.20 meters
2b. actual distance = 17.65 meters
3. \( \% \text{ error} = \left( \frac{0.45}{17.65} \right) \times 100 = 2.5 \)
4. No. An apparent angular size of \( \frac{1}{10} \) radian indicates 10 body lengths between observer and object. Unless both lab partners have the same height, this distance in meters will vary. It will be closer when the "object person" is short, and farther away when the "object person" is tall.

**TEST FOR UNDERSTANDING**

Q: How far away is a 6 foot person standing if they subtend \( 2^\circ \) in your visual field?

A: \( 2^\circ = \frac{1}{28.7} \) rad = \( \frac{d}{D} \) = 6 ft/D

\[ D = 6 \text{ ft} \times 28.7 = 172 \text{ feet} \]
**E5: Knowing DISTANCE, estimate height.**

1. Work in a hallway, gym or outside with the same lab partner as in activity E4.
   - Stand 30 meters apart.
   - Accurately measure and record each other’s apparent height in radians using your Kamals.

2. Calculate your lab partner’s actual height in meters. Write a proportion:
   \[
   \frac{1}{x \times x} \text{rad} = \frac{d_{\text{partner's height}}}{D_{\text{distance to partner}}}
   \]

3. How close did you come to the actual height of your lab partner measured in Lab E4.
   \[
   \% \text{ error} = \frac{\text{difference}}{\text{actual height}} \times 100
   \]

**Lesson Notes**

1b. The Kamal will not likely be prefolded at the specific angle that most accurately matches each lab partner’s apparent height. Advise students to first locate an approximate angular height, (perhaps \(\frac{1}{17}\) rad for many students in this experiment), then roll the triangle between thumb and index finger so it folds continuously through a range of readings above and below this approximation.

**Model Answers**

Answers will vary. Here is one possible set of calculations:

1a. Actual distance to lab partner in meters = \(D = 30\) meters

1b. Apparent angular height of lab partner = \(\frac{1}{17.2}\) rad

2. Calculated height of lab partner in meters:
   \[
   \frac{1}{17.2} = \frac{d}{D} = \frac{d}{30}
   \]
   \[d = \frac{30}{17.2} = 1.74\ m\]

3. Actual height of lab partner = 1.72 m
   \[
   \% \text{ error} = (0.02/1.72) \times 100 = 1.2
   \]

**Test for Understanding**

Q: How tall is a person who subtends 3° in your visual field at a distance of 100 feet?

A: \[3° = \frac{1}{19.1}\text{ rad} = \frac{d}{D} = \frac{d}{100\ ft}\]
   \[d = 100\ ft/19.1 = 5.24\ feet \approx 5\ feet\ 3\ inches\]

**Extension**

The original Kamal was a rectangular wooden bar of fixed length. Its distance from the eye was changed by holding a knotted string between the teeth; different knots changed the distance of the Kamal from the eye. Interested students may wish to design their own Kamals based on knotted string. A diary of the invention process would be a superb enrichment project, with a successful design making a fine class demonstration.

Considerations to ponder:
- What length of string: 50 cm? 100 cm?
- What to tie on the end: craft stick? paper clip?
- Where to position knots: Halving distance doubles the subtended angle.
- How to accurately position knots: Put a straight pin in the loop. Pull it to where you want to place the knot.
Module E: TIE YOUR KAMAL

E6: Lay out the inner solar system plus Jupiter.

1. Get the Planets to Scale page.
   a. Calculate (on notebook paper), the angular size of the Sun as observed from each planet in d/D rad. Simplify to 1/n rad, then record each result in the box for that planet.
   b. Use a hand lens to read the smallest calibrations on your Kamal. What do you notice?

2. Cut out the “Sun” patch. Tape it to the back of a paper plate so its curved edge meets the rim.
   a. Measure the diameter of this paper-plate Sun to the nearest 0.01 meters. Record it in the box provided.
   b. Cut into the plate along each side of this patch. Slip the flap into a small jar so the “Sun” stands upright. (Tape in place if needed.)

3. Cut and tape the planet patches. “Jupiter” may require extra tape so it doesn’t flop over.

4. Gather these items in a box or bag to take outside: model planets and Sun, Kamal, 10-meter string, lab instructions, paper and pencil, calculator, clipboard (optional).

5. Lay out this model Solar System along a sidewalk, road, track or field. Use your Kamal to place each planet far enough back from the “Sun” to match the apparent size you calculated.

6. Use your 10 meter string to evaluate how accurately you placed Earth. (Check out other planets if you have time.)

7. Turn the planets so you can see each one from “Earth.” Compare their visibility (apparent sizes) to what you see in the night sky.

LESSON NOTES

1. Convert d/D to the form 1/n by dividing numerator and denominator by d. If figures overrun calculator capacities, divide all values by 1000 (knock off 3 zeros).

5. If you have solar-safe mylar, check that this paper-plate Sun correctly models the real thing! Working in pairs, one student can block the Sun while the other confirms that the paper-plate and real Sun have the same apparent size as viewed from Earth (1/108 rad).

   Notice that a Moon Ruler perfectly frames the paper-plate “Sun” from “Earth.” For those observers lucky enough to witness a full solar eclipse, this happy coincidence allows a spectacular view of the Sun’s corona!

MODEL ANSWERS

1a. Mercury: 1,390,000/57,900,000 = 0.0240 rad = 1/42 rad
    Venus: 1,390,000/108,000,000 = 0.0129 rad = 1/76 rad
    Earth: 1,390,000/150,000,000 = 0.00927 rad = 1/108 rad
    Mars: 1,390,000/228,000,000 = 0.00609 rad = 1/164 rad
    Jupiter: 1,390,000/778,900,000 = 0.00179 rad = 1/680 rad

1b. Tiny calibrations on the Kamal’s tip specifically match the angular size of the Sun as seen from Mars and Jupiter.

2a. Diameter of paper-plate Sun = 0.23 meters.

6. If the Sun’s apparent size is 1/108 rad as viewed from Earth, they are separated in this scale model by:
   108 plates x 0.23 meters/plate = 24.8 meters.

   Expect students to come within perhaps ±2.5 meters of this value (a 10% placement error). Applied to far-out Jupiter, this 10% error works out to ±13 meters.

7. Mercury (not visible), Mars (a tiny speck) Jupiter (easy to see), Venus (easiest). This relative ranking corresponds to the real planets we see at night as they orbit near Earth. Reflecting sunlight in a dark night sky, these planets are more visible than in this daylight model.

   Angular sizes are also interesting to compare. Mercury @ 1/10,270 rad is roughly 10 times smaller than Jupiter @ 1/927 rad (slightly larger than a 1 mm dot at 1 meter.)

TEST FOR UNDERSTANDING

Q: Earth (diameter = 12,760 km) approaches to within 128,000,000 km of Jupiter when both planets are in alignment. How big does Earth look from Jupiter at closest?

A: 12,760/128,000,000 = 1/10,000 rad. Earth appears as a tiny point of light near the Sun, similar to Mercury’s apparent size as seen from Earth.
E7: Do other planets have larger moon-rises?

Is there any planet in our solar system that has a rising moon that looks apparently larger than our own Luna?
(Assume clear views, unimpeded by atmosphere or darkness.)

1. What important variables must you research?
2. Report your results.

research: E1 E2 E3 E4 E5 E6 E7
materials: calculator, research materials from a library or internet access.
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LESSON NOTES
1. Key words for a Google search:
   moons, solar system, data tables.
2. All moon dimensions stated in radii must be converted to diameters.

TEST FOR UNDERSTANDING
Q: Estimate Jupiter’s apparent angular size as seen from its moon Io. Imagine how this might look in terms of paper plates.
A: Jupiter-Io distance = 421,600 km
   Jupiter’s diameter = 138,150 km
   d/D = 138,150/421,600 = \(\frac{1}{3}\) rad \(\approx 19^\circ\)

   Jupiter looms very large on Io’s horizon, similar to the size of a paper plate (d) viewed from a distance of 3 paper plates (D). d/D = \(\frac{1}{3}\) rad.

   Jupiter actually subtends a somewhat smaller angle, just as a coin on the Coin Ruler subtends somewhat less than \(\frac{1}{3}\) rad.

MODEL ANSWERS
1. Variables to research: moon diameters in kilometers (d) and distance from the home planet in kilometers (D). The largest moonrise in our solar system will be the largest subtended angle in radians (d/D).
2. Here is a listing of moons ranked by apparent angular size d/D rad, as seen from their home planets. All distances are in kilometers:
   - Pluto’s Charon: 1,200/19,640 = 1/16 rad
   - Earth’s Luna: 3,476/384,400 = 1/110 rad
   - Jupiter’s Io: 3,629/421,600 = 1/116 rad
   - Uranus’ Ariel: 1160/191,240 = 1/165 rad
   - Jupiter’s Ganymede: 5,276/1,070,000 = 1/203 rad
     (largest moon in our solar system)
   - Jupiter’s Europa: 3,126/670,900 = 1/215 rad
   - Uranus’ Umbriel: 1190/265,970 = 1/224 rad
   - Saturn’s Titan: 5150/1,221,850 = 1/237 rad
     (second largest moon in our solar system)

   With the exception of Pluto’s Charon, Earth’s Luna appears largest, barely bigger than Jupiter’s Io. If we consider Pluto a planet rather than an asteroid, and if darkness were not an impediment, then Charon would appear immense when viewed from Pluto, almost 7 times larger than Luna viewed from Earth. It is questionable, however, with the feeble light reaching Charon from the distant Sun, whether it would be visible to the naked eye. It would eclipse background stars, however, and might be observed indirectly in this manner.
Module F: LOOK FROM HERE and THERE

Moving TOWARD the circumference...

Picture your eye as the center of a circle, with a tree along the circumference: As you move toward it, changes in Kamal readings at (a) and (b) allow you to calculate its actual HEIGHT.

Moving ALONG the circumference...

Now picture the tree at the center of a circle as your eye moves along its circumference: a Kamal reading of its parallax shift from (A) to (B) against a distant background, allows you to calculate its actual DISTANCE.

MAP YOUR LABS: Labs F1-F4 require the Kamal constructed in Module E. Lab F3 is an especially important introduction to parallax that anchors labs F4-F6 that follow.

F1: Concept
Estimate diameter from a distance.
Take a Kamal reading. Move closer and take another reading. Solve simultaneous equations for the diameter of a "remote" paper plate.
(Time: 30 minutes)

F2: Application
Measure something tall three ways.
Find something tall outside. Estimate its height with your Kamal and clever math following three different strategies.
(Time: 50 minutes)

F3: Concept
What's parallax, and how can we measure it?
Watch the end of a meter stick "jump" back and forth relative to distant landmarks. Measure this parallax with your Kamal and with a ruler.
(Time: 50 minutes)

F4: Application
Step aside and look again.
Set up a 1-meter baseline, positioned sideways to a "remote" tree or pole. Measure its parallax with your Kamal and calculate its distance.
(Time: 40 minutes)

F5: Extension
How far to nearby stars?
Knowing the parallax shift of nearby Sirius against distant background stars, estimate how far away it is in astronomical units and light years.
(Time: 20 minutes)

F6: Extension
I beg your parsec...
Understand how a distance of one parsec is defined. Use this definition to calculate the distance to Sirius in parsecs.
(Time: 30 minutes)
F1: Estimate diameter from a distance.

1. Tape a paper plate to a wall.
2. Suspend a meter stick at eye level as follows:
   a. Wedge the fiber end of a broom head between the seat and back of a chair. Secure the handle with strong tape.
   b. Center a meter stick across the broom handle, parallel to the chair back and the floor. Hold it there with strong tape looped over the top.
3. Position this chair 4 or 5 walking paces from the paper plate. Turn the chair to aim the meter stick at the plate.
4. Use your Kamal to estimate the angular size of the paper plate from both ends of the meter stick:
   a. Roll the Kamal between thumb and finger so you can fold it easily anywhere above or below your range of angles.
   b. Measure over several trials to the nearest tenth of the object distance. Squinting improves accuracy!
5. Call the plate's distance to one end of the meter stick D meters. How many meters to the other end in terms of D?
   a. Express the plate diameter d and both observing distances in terms of the radians you measured.
   b. You've got two equations and two unknowns. Solve for D.
6. Measure the paper plate directly to estimate your percent error. \[ \text{% error} = \frac{\text{difference} \div \text{actual}}{\text{actual}} \times 100 \]

concept: E's F1 F2 F3 F4 F5 F6
materials: paper plate, scissors, packaging tape, broom, chair, meter stick, Kamal, a calculator

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F2: Measure something tall three ways.

1. Find something tall with clear views to its base (no tall grass in the way) and to its top (easy to see against contrasting background). Calculate its height 3 ways:
   a. Stand far enough away so its height precisely matches 1/8 rad on your Kamal. Stand 10.0 meters further away and measure its angular height again.
   b. Move in closer so that 1/8 rad fits precisely twice. Measure the intervening distance with your 10-meter string and a meter stick.
   c. Stand back again so 1/8 rad fits top to bottom. Measure the intervening distance.
2. What is your best estimate of your object's actual height? Explain your reasoning.

application: E's F1 F2 F3 F4 F5 F6
materials: 10 meter string, Kamal, meter stick, clip board (optional), calculator

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F3: What's parallax, and how can we measure it?

1. Bend a paper clip to a right angle. Tape it at 100 cm on a meter stick.

   a. Hold 0 cm directly under either eye. Aim the stick so your eye a sights some spot A on your distant horizon through paper clip p.
   b. Without moving the stick, simply close eye a and open eye b. Notice how object p appears to “jump” so it aligns with a different spot B on your horizon.

concept: E's F1 F2 F3 F4 F5 F6

materials: paper clip, Kamal, meter stick, masking tape, mirror (optional), calculator

2. This apparent movement of the paper clip against fixed, distant objects is called parallax. Does paper clip p actually shift as you change your point of view? Explain.

3. The amount of parallax depends on the distance between viewing points (in this case, your eyes) as well as the distance to the viewed object (in this case, the paper clip).

   a. Scan the horizon to find 2 distinctive landmarks, aligning with paper clip p as you close each eye.

   b. Use your Kamal to measure the apparent jump (the parallax ApB), in radians.

4. Notice that the distance ab (between your eyes) is part of the circumference of a circle, centered at paper clip p, with a one meter radius ap (the length of the meter stick).

   a. Measure baseline ab, eye pupil to eye pupil. Use a wall mirror or ask a lab partner to help.
   b. Calculate angle apb in radians: $\text{apb} = \frac{d}{D}$
   c. Compare this calculation to your Kamal reading in step 3b.

5. If you change distance ap to 75 cm, will that change the measured amount of parallax shift? Illustrate your answer with a diagram. Take new measurements if time allows.

F4: Step aside and look again!

1. Locate a straight, vertical foreground object outside that...

   a. ...meets or crosses your horizon. (A flagpole, tree top, edge of a tall building all serve.)
   b. ...has 10 to 100 meters of intervening clear space that you can walk across and measure.

2. Set up a meter stick/broom/chair arrangement as outlined in activity F1.

   a. Imagine yourself standing on the circumference of a circle with your foreground object at the center.
   b. Place the meter stick sideways, into the circumference of this circle.

   B A

3. Estimate the distance to your foreground object...

   a. ...using parallax and your Kamal.
   b. ...measuring with a sting.

4. Compute your percent error.

application: E's F1 F2 F3 F4 F5 F6

materials: chair, broom, packaging tape, meter stick, 10-meter string, a Kamal, clipboard (optional)

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Module F: LOOK FROM HERE and THERE

F5: How far to nearby stars?

1. Nearby Sirius, the brightest star in our night sky, appears to shift 0.000001818 radians off center and back again (relative to background stars) through a 6-month cycle. For the other half year, it continues in the opposite direction to complete a tiny oval.

   a. What's going on?
   b. How many Astronomical Units (AUs) is Sirius from our solar system?
      (Hint: Convert to a fractional radian with 1 in the numerator.)
   c. Multiply by unit conversion factors to convert this distance to kilometers; to light years.

   extension: F1 F2 F3 F4 F5 F6

   materials: a calculator

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F6: I beg your parsec...

1. Parallax causes nearby stars to apparently shift in relation to distant background stars in telescopic photographs.

   a. Why do these apparent motions all have periods equal to 1 year?
   b. How do they trace these shapes?

   extension: F1 F2 F3 F4 F5 F6

   materials: answers from F5 and a calculator

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2. If a nearby star's maximum 3-month shift equals 1 arc second, then (by definition) it is a distance of 1 parsec away.

   a. How many AUs distant is a 1-parsec star from our solar system?
      (Hint: Convert 1 arc second to a fractional radian with 1 in the numerator.)
   b. Is Sirius closer or farther away than 1 parsec? Explain.
   c. Make a basic calculation to find how many light years are in 1 parsec.

   An ARC SECOND is how wide a single human hair would look from 12 meters away!
**F1: Estimate diameter from a distance.**

1. Tape a paper plate to a wall.
2. Suspend a meter stick at eye level as follows:
   a. Wedge the fiber end of a broom head between the seat and back of a chair. Secure the handle with strong tape.
   b. Center a meter stick across the broom handle, parallel to the chair back and the floor. Hold it there with strong tape looped over the top.
3. Position this chair 4 or 5 walking paces from the paper plate. Turn the chair to aim the meter stick at the plate.
4. Use your Kamal to estimate the angular size of the paper plate from **both** ends of the meter stick:
   a. Roll the Kamal between thumb and finger so you can fold it easily anywhere above or below your range of angles.
   b. Measure over several trials to the nearest tenth of the object distance. Squinting improves accuracy!
5. Call the plate's distance to one end of the meter stick D meters. How many meters to the other end in terms of D?
   a. Express the plate diameter d and both observing distances in terms of the radian angles you measured.
   b. You've got two equations and two unknowns. Solve for D.
6. Measure the paper plate directly to estimate your percent error. \[ \% \text{error} = \left(\frac{\text{difference}}{\text{actual}}\right) \times 100 \]

**concept:** E's F1 F2 F3 F4 F5 F6

**materials:** paper plate, scissors, packaging tape, broom, chair, meter stick, Kamal, calculator

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**LES SSON NOTES**

2a. Use packaging tape to firmly hold the broom and meter stick. Masking tape isn’t strong enough. If your chairs don’t have an opening in the back, borrow folding chairs. Two or three chair-broom assemblages can serve many students.

4b. Squinting creates a "pinhole" effect that sharpens, and therefore lengthens, the Kamal’s edge as your eye matches it to the more distant plate. You can observe this effect by holding the back of your thumb near one eye so its blurry form blocks the bottom half of your visual field. Now squint. Notice how your thumb "grows" in size as it comes into better focus!

Students working with lab partners might take independent Kamal readings and compare results.

**MODEL ANSWERS**

5. If the far end of the meter stick is D meters from the plate, then the near end is D - 1. Likewise, if the near end of the meter stick has a distance of D meters, then the far end is D + 1. Either way, you get two equations that yield the same result. Pre-algebra students will need special help.

5a. Here are model calculations based on a 9 inch (0.229 m) paper plate:
   - far end of meter stick: Kamal reads 1/16.1 rad at a distance D.
   - near end of meter stick: Kamal reads 1/11.9 rad at a distance D - 1.
   - far end equation: \[ \frac{d}{D} = \frac{1}{16.1} \]; thus \[ D = 16.1 \text{ d} \]
   - near end equation: \[ \frac{d}{(D - 1)} = \frac{1}{11.9} \]; thus \[ 11.9 \text{ d} = D - 1 \].
   - by substitution: \[ 11.9 \text{ d} = 16.1 \text{ d} - 1 \]; thus \[ 4.2 \text{ d} = 1 \]. So \[ d = 0.238 \text{ m} \].

6. \[ \% \text{error} = \left(\frac{0.009 \text{ m}}{0.229 \text{ m}}\right) \times 100 = 3.9 \%

**TEST FOR UNDERSTANDING**

Q: A lab partner standing in a field has an apparent angular size of 1/10 rad. You take 1 giant step forward (3 feet) and she "grows" to 1/9.5 rad in your visual field. How tall is she?

A: \[ \frac{d}{D} = \frac{1}{10} \]; thus \[ D = 10 \text{ d} \].

- \[ \frac{d}{(D - 3)} = \frac{1}{9.5} \]; thus \[ 9.5 \text{ d} = D - 3 \].
- by substitution: \[ 9.5 \text{ d} = 10 \text{ d} - 3 \].
- \[ 0.5 \text{ d} = 3 \]; so \[ d = 6 \].

So your lab partner is 6 ft tall.
**F2: Measure something tall three ways.**

1. Find something tall with clear views to its base (no tall grass in the way) and to its top (easy to see against contrasting background). Calculate its height 3 ways:
   - a. Stand far enough away so its height precisely matches 1/8 rad on your Kamal. Stand 10.0 meters further away and measure its angular height again.
   - b. Move in closer so that 1/8 rad fits precisely twice. Measure the intervening distance with your 10-meter string and a meter stick.
   - c. Stand back again so 1/8 rad fits top to bottom. Measure the intervening distance.

2. What is your best estimate of your object's actual height? Explain your reasoning.

**Lesson Notes**

1a. Scout out in advance a tall object for students to measure, such as the face of a building, a tree, telephone pole or similar object. There must be sufficient space to step back more than 8 times the height of this object, while maintaining clear, distinct views to its base and top.

**Model Answers**

For each calculation a-c below, (d) represents the height of a telephone pole, and (D) stands for the distance you are standing from its base.

1a: Near end of 10 m string: $d/D = \frac{1}{8}$ rad. So, $D = 8d$
Far end of 10 m string: $d/(D+10) = \frac{1}{8}$ rad. So, $D + 10 = 9.1d$
By substitution: $8d + 10 = 9.1d$. So, $1.1d = 10, and d = 9.1$
The telephone pole is 9.1 meters tall.

1b: Angular height of 1/2 pole = 1/8 rad
Angular height of full pole = 1/8 rad + 1/8 rad = 1/4 rad
$D = 35.3$ meters
$d/D = 1/4$ rad = $d/35.3$. So, $d = 8.8$.
The telephone pole is 8.8 meters tall.

1c: $d/D = \frac{1}{10}$ rad. So $D = 8d$
Distance to pole = $D = 74.5$ meters
$d/D = 1/8$ rad = $d/74.5$. So, $d = 9.3$
The telephone pole is 9.3 meters tall.

2. This question is open ended, inviting a range of response. Short of climbing the pole to measure it directly, method c gives the most accurate estimate of the telephone pole's true height for several reasons:

Calculations in method c are based on just one estimate of angular height, unlike methods a and b that rely on two estimates with uncertainties adding together.

**Application:** E's F1 F2 F3 F4 F5 F6

**Materials:** 10 meter string, Kamal, meter stick, clip board (optional), calculator

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Ground distance measured in method c is twice as long as b, and many times longer than a, so that measuring uncertainty is a smaller percentage of the whole length.

Running a small difference in estimated angular height through method a calculations (1/9.0 rad and 1/9.2 rad), results in significant changes in estimated height (10.0 m and 8.3 m).

Notice on the Coin Ruler that the straight version of a coin 1/4 rad tall subtends a slightly smaller angle than its curved shape. Similarly, a straight telephone pole observed from a distance of 4 poles subtends 1/4.08 rad, an angle slightly smaller than 1/4.00 rad for a slightly curving pole.

$\tan^{-1} 1/4.00 = 14.036^\circ$

$14.036^\circ \times \pi \text{ rad} / 180^\circ = 0.245 \text{ rad} = 1/4.08 \text{ rad}$

**Extension**

Plan an experiment to calculate the height of a mountain. Extra credit if you pull it off.

**Test for Understanding**

Q: Your ship is steaming away from an island paradise with a snow-capped mountain rising majestically skyward. After an hour’s travel, this mountain finally comes within the 1/8 radian range of your Kamal. You sail 4 miles farther and take a second reading of the mountain, now at 1/10 radian. Estimate its actual height.

Far view: $d/(D+4) = 1/10$. So, $D + 4 = 10d$.
By substitution: $8d + 4 = 10d$. So, $d = 2$.
The mountain is 2 miles high.
**F3: What's parallax, and how can we measure it?**

1. Bend a paper clip to a right angle. Tape it at 100 cm on a meter stick.

   a. Hold 0 cm directly under either eye. Aim the stick so your eye a sights some spot A on your distant horizon through paper clip p.

   b. Without moving the stick, simply close eye a and open eye b. Notice how object p appears to “jump” so it aligns with a different spot B on your horizon.

   concept: E's F1 F2 F3 F4 F5 F6

   materials: paper clip, Kamal, meter stick, masking tape, mirror (optional), calculator

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2. This apparent movement of the paper clip against fixed, distant objects is called **parallax**. Does paper clip p actually shift as you change your point of view? Explain.

3. The amount of parallax depends on the distance between viewing points (in this case, your eyes) as well as the distance to the viewed object (in this case, the paper clip).

   a. Scan the horizon to find 2 distinctive landmarks, aligning with paper clip p as you close each eye.

   b. Use your Kamal to measure the apparent jump (the parallax ApB), in radians.

4. Notice that the distance ab (between your eyes) is part of the circumference of a circle, centered at paper clip p, with a one meter radius ap (the length of the meter stick).

   a. Measure baseline ab, eye pupil to eye pupil. Use a wall mirror or ask a lab partner to help.

   b. Calculate angle apb in radians:

   \[
   \text{apb} = \frac{d}{D}
   \]

   c. Compare this calculation to your Kamal reading in step 3b.

5. If you change distance ap to 75 cm, will that change the measured amount of parallax shift? Illustrate your answer with a diagram. Take new measurements if time allows.

---

**INTRODUCTION**

Hold up your thumb, arm outstretched. Close one eye, then the other. Notice how your thumb appears to jump back and forth relative to background objects. This apparent movement of foreground against background is called parallax. Nothing really jumps except the eye you look through.

**LESSON NOTES**

1. Students who can’t close each eye independently can use a hand to cover one eye at a time.

   If your room has a window with a view to a reasonably distant horizon, your students may not need to go outside for this experiment.

3. Landmarks A and B must be distinct. If you can’t turn away then find them again, or accurately measure the angular distance between them with your Kamal, then find a less ambiguous pair.

**MODEL ANSWERS**

2. No. The paper clip and meter stick remain stationary. Only my point of view (my eye) changes.


4a. Baseline eye separation = 70 mm.

4b. Angle apb = d/D radians = 70 mm / 1000 mm = 0.0700 rad = 1/14.3 rad.

4c. The parallax shift observed with the Kamal (1/14.0 rad), is reasonably close to the calculated angle d/D (1/14.3 rad).

5. Moving the paper clip closer, from 100 cm to 75 cm, increases its parallax shift against a distant background. (The observed and calculated radian angles become 25% larger.)

**TEST FOR UNDERSTANDING**

Q: Your extended thumb appears to jump 1/10 rad between fixed distant landmarks as you alternately close one eye and then the other. If your reach is 70 cm, how widely set are your pupils?

A: d/D = 1/10 = d/70; thus, 10 d = 70, and d = 7.

Your eyes are 7 cm apart.
F4: Step aside and look again!

1. Locate a straight, vertical foreground object outside that...
   a. ...meets or crosses your horizon. (A flagpole, tree top, edge of a tall building all serve.)
   b. ...has 10 to 100 meters of intervening clear space that you can walk across and measure.

2. Set up a meter stick/broom/chair arrangement as outlined in activity F1.
   a. Imagine yourself standing on the circumference of a circle with your foreground object at the center.
   b. Place the meter stick sideways, into the circumference of this circle.

3. Estimate the distance to your foreground object...
   a. ...using parallax and your Kamal.
   b. ...measuring with a sting.

4. Compute your percent error.

**application**: E's F1 F2 F3 F4 F5 F6

**materials**: chair, broom, packaging tape, meter stick, 10-meter string, a Kamal, clipboard (optional)

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**LESSON NOTES**

1. Scout out a suitable foreground object in advance for students to measure. Its background horizon should be at least 10 times more distant than the object itself, more if possible. Trees on a distance ridge offer rich landmark possibilities.

2. In parallax diagrams the observer's baseline (d) is on the circumference of the circle, while the distance (D) to the object of interest is at the circle's center. The subtended angle, as always, is d/D radians.

   This parallax experiment models applications in astronomy. We are estimating the distance to a nearby tree (nearby star) by measuring its parallax shift against landmarks A and B on a fixed horizon (fixed background stars). Our baseline is 1 meter (astronomers use 1AU, the average distance between Earth and Sun). We measure the parallax shift using our Kamal (astronomers use telescopic photographs).

**MODEL ANSWERS**

3. Angular distance between landmark objects A and B:
   - Kamal measurement: 1/50 rad; Baseline: 1 meter
   - \[ \frac{d}{D} = \frac{1}{50} = \frac{1}{D} \]
   - thus, D = 50 meters
   - Students should confirm their parallax calculation by measuring with the 10 meter string. If they come within 10% error, they are doing well.

**TEST FOR UNDERSTANDING**

Q: Shifting your head 1 meter to the left makes a distant flagpole appear to shift 1/100 rad to the right against a distant fixed background. How far away is the flag pole?

A: \[ \frac{d}{D} = \frac{1}{100} = \frac{1}{D}; \text{ thus, } D = 100. \]

The flagpole is 100 meters away.
F5: How far to nearby stars?

1. Nearby Sirius, the brightest star in our night sky, appears to shift 0.000001818 rads off center and back again (relative to background stars) through a 6-month cycle. For the other half year, it continues in the opposite direction to complete a tiny oval.

   a. What's going on?
   b. How many Astronomical Units (AUs) is Sirius from our solar system?
      (Hint: Convert to a fractional radian with \( \pi \) in the numerator.)
   c. Multiply by unit conversion factors to convert this distance to kilometers; to light years.

   ![Diagram showing Earth's path around Sun and Sirius star's apparent movement against distant stars.]

   Apparent movement against distant stars

   Nearby STAR

   (Not to scale!)

   | extension: F1 F2 F3 F4 F5 F6 | materials: a calculator |
   ---|---|---

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LESSON NOTES

1. The apparent motion of a nearby star against far-away background stars (its parallax shift), is very small. Expressed in decimal radians, the numerous zeros following the decimal point will exceed the capacity of 8-digit display calculators. One easy way out is to provide 10-digit display calculators or scientific calculators for student use. A more educational strategy is to teach the math that will enable students to use the more limited 8-digit displays with understanding. Frustration with display overruns turns into a teachable moment about scientific notation, or the algebraic manipulation of fractions.

   If you are old enough to have grown up using slide rules, you know that they provided the digits, usually only 3 or 4 significant figures, but not the decimal point. This limitation was an educational boon in disguise. We had to learn to place the decimal in order to use that instrument. The same situation applies here with 8-digit calculators.

   If you would like your class to improvise their own slide rules from folded paper, and help students understand scientific notation, logarithms, number estimation and such, order FAR OUT MATH #43 from our catalog.

MODEL ANSWERS

1a. Parallax happens! What’s really moving is us (our Earth), causing an apparent shift in nearby Sirius as we move first to one side of our Sun and then to the other side in our yearly orbit.

1b. \[ 0.000001818 = 1.818 \times 10^{-6} \]

   \[ = 1.818 \times 10^{-6} + 1.818 \times 10^{-6} / 1 + 1.818 \times 10^{-6} \]

   \[ = 1/550,055 = d/D \text{ rad} \]

   thus, Sirius is 550,055 AU from our solar system.

1c. \[ (550,055 \text{ AU}) \times (150,000,000 \text{ km} / \text{AU}) \]

   \[ = 5.50055 \times 10^5 \text{ km} / 1 \times 10^8 \text{ km} \]

   \[ = 8.25 \times 10^{13} \text{ km} \]

   \[ (8.25 \times 10^{13} \text{ km} / 1) \times (1 \text{ ly} / 9.46 \times 10^{12} \text{ km}) \]

   \[ = 8.25 \times 10^{13} \text{ km} / 9.46 \times 10^{12} \text{ km} \]

   \[ = 0.87 \times 10^1 \text{ ly} = 8.7 \text{ ly} \]

TEST FOR UNDERSTANDING

Q: Alpha Centauri, our nearest star neighbor, is 268,000 Astronomical Units from our solar system. What is its parallax?

A: \[ \text{parallax} = d/D = 1/268,000 \text{ rads} \]
F6: I beg your parsec...

1. Parallax causes nearby stars to apparently shift in relation to distant background stars in telescopic photographs.

   a. Why do these apparent motions all have periods equal to 1 year?

   b. How do they trace these shapes?

   LINE
   OVAL
   CIRCLE

2. If a nearby star’s maximum 3-month shift equals 1 arc second, then (by definition) it is a distance of 1 parsec away.

   a. How many AU's distant is a 1-parsec star from our solar system?
      (Hint: Convert 1 arc second to a fractional radian with 1 in the numerator.)

   b. Is Sirius closer or farther away than 1 parsec? Explain.

   c. Make a basic calculation to find how many light years are in 1 parsec.

extension: F1 F2 F3 F4 F5 F6
materials: answers from F5 and a calculator

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MODEL ANSWERS

1a. The Earth moves around the Sun in a 1-year cycle. This real motion causes nearby stars to apparently move through parallax shifts of equal 1-year periods.

1b. Nearby stars in the plane of our solar system appear to move back and forth in a straight line. Nearby stars perpendicular to the central axis of our solar system appear to move in a perfect circle. Between these two special cases is a range of ovals that range from thin to nearly circular.

2a. 

\[(1 \text{ arc sec}/1) \times (1 \text{ arc min}/60 \text{ arc sec}) \times (1/60 \text{ arc min}) \times (\pi \text{ rad}/180^\circ)\]

\[= \pi/648,000 \text{ rad}\]

\[= \pi + \pi / 648,000 + \pi\]

\[= 1/206,265 \text{ rad}\]

Thus, a 1 parsec star is 206,265 AU distant from our solar system.

2b. The parallax of Sirius (1/550,055 rad) is a smaller angle than the parallax of a one-parsec star (1/206,265 rad). Less parallax shift means Sirius is farther away.

2c. In Lab F5, Sirius was calculated to be 8.7 ly distant. So a 1-parsec star with a larger radian shift must be less than this. That is, 8.7 ly is proportionally multiplied by a number less than one:

\[8.7 \text{ ly} \times 206,265/550,055 = 3.26 \text{ ly}\]

LESSON NOTES

2. Par is a Latin prefix meaning equal. Hence "parsec" means "equal to 1 second of arc." Parsec is abbreviated "pc."

EXTENSION

A human hair is about 50 micrometers wide (60x10^-6 m). Show that its subtended angle at 12 meters is about 1 arc second.

\[d/D = 60 \times 10^{-6}/12 \times 10^0 = 5 \times 10^{-6} \text{ rad}\]

\[= (5 \times 10^{-6} \text{ rad}/1) (180^\circ/\pi \text{ rad}) (60 \text{ arc min}/1^\circ)\]

\[= 1/206,265 \text{ rad} \approx 1 \text{ arc second}\]

TEST FOR UNDERSTANDING

Q: If a star that shifts 1 arc second is 1 parsec away, how many parsecs distant is a star that shifts...

1/2 arc sec?
1/4 arc sec?
1/10 arc sec?
1/100 arc sec?

A: 1/2 arc sec shift = star that's 2 pc distant
1/4 arc sec shift = star that's 4 pc distant
1/10 arc sec shift = star that's 10 pc distant
1/100 arc sec shift = star that's 100 pc distant